# How to Make a Decision under Interval Uncertainty If We Do Not Know the Utility Function

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**Formulation of the problem.** According to decision theory, decisions of a rational person are described by a function $u(a)$ called *utility*: an alternative in which we gain amount $a$ is better than the alternative in which we gain amount $b$ if and only if $u(a) > u(b)$. For the case of gain, the utility function is (non-strictly) increasing: if $a \le b$ then $u(a) \le u(b)$.

In practice, we only know the consequence of each action with uncertainty. In many cases, all we know is the bounds on possible gain, i.e., the interval $[\underline{a}, \overline{a}]$ of possible values of the gain.

In this case, according to decision theory, the decision maker should select some value $\alpha \in [0, 1]$ describing the decision maker's degree of optimism-pessimism – and select an alternative for which the value $\alpha \cdot u(\overline{a}) + (1 - \alpha) \cdot u(\underline{a})$ is the largest.

Sometimes, we do not know the utility function. When can we still conclude that $[\underline{a}, \overline{a}]$ is better than $[\underline{b}, \overline{b}]$? The answer is easy when $\alpha = 1$ – then we select the alternative with the larger $\overline{a}$ – and when $\alpha = 0$ – then we select the alternative with the larger $\underline{a}$. But what if $0 < \alpha < 1$?

**Main result.** *For every $\alpha \in (0, 1)$ and for every two intervals $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$, the following two conditions are equivalent:*

1. $\alpha \cdot u(\overline{a}) + (1 - \alpha) \cdot u(\underline{a}) \ge \alpha \cdot u(\overline{b}) + (1 - \alpha) \cdot u(\underline{b})$ *for all non-strictly increasing functions $u(a)$;*
2. $\underline{a} \ge \underline{b}$ *and* $\overline{a} \ge \overline{b}$.

**Proof.** Condition 2. implies condition 1. due to monotonicity. Let us prove that if the condition 2. is not satisfied, i.e., if $\underline{a} < \underline{b}$ or $\overline{a} < \overline{b}$, then the condition 1. is violated for some increasing function $u(a)$.

Indeed, if $\underline{a} < \underline{b}$, then we can take $u(a) = 0$ for $a \le \underline{a}$ and $u(a) = 1$ otherwise. Then, since $\overline{b} \ge \underline{b} > \underline{a}$, the right-hand side of the inequality 1. is equal to $\alpha + (1 - \alpha) = 1$, while the left-hand side is equal to $\alpha \cdot u(\overline{a}) \le \alpha < 1$ – so the inequality is not satisfied.

If $\overline{a} < \overline{b}$, then we can take $u(a) = 0$ for $a < \overline{b}$ and $u(a) = 1$ otherwise. Then, since $\underline{a} \le \overline{a} < \overline{b}$, the left-hand side of 1. is equal to 0, while the right-hand side is larger than or equal to $(1 - \alpha) \cdot u(\overline{b}) = 1 - \alpha > 0$, so the inequality is not satisfied either. The result is proven.

*Comment.* We considered the case when we know $\alpha$ and but we do not know $u(a)$. Similar results can be proven in two other cases:

- when we know $u(a)$ – which is strictly increasing – but we do not know $\alpha$;
- when we do not know neither $\alpha$ nor $u(a)$.

For example, when we do not know $\alpha$, then we can have $\alpha = 0$ and $\alpha = 1$ – in which case we also have $\underline{a} \ge \underline{b}$ and $\overline{a} \ge \overline{b}$.