Paradox of Causality and Paradoxes of Set Theory

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Paradox of causality: reminder. It is well known that tine travel can lead to paradoxes: if a person A travels to the past and kills his own grandfather before his own father was conceive, there is no possibility for A to be born – but he was actually born. In general, this paradox appears every time we close a closed loop of causality relations:

- whether we have event e_1 causally affectibg event e_2 (we will denote it by $e_1 < e_2$) and e_2 causally affecting event e_1 ($e_2 < e_1$),
- or whether we have $e_1 < e_2$, $e_2 < e_3$, and $e_3 < e_1$,
- on whether we have a even longer loop.

Paradox of intuitive ("naive") set theory. In set theory, a known paradox – first discovered by Bertran Russell – is related to the possibility of having a simple element-of loop, i.e., the possibility to have $x \in x$ for some set x. Specifically, the paradox appears when we consider the set $S = \{x : x \notin x\}$ of all the sets that are not elements of themselves. The paradox appears when we check whether $S \in S$. Indeed, we have either $S \in S$ or $S \notin S$, and in both cases, we get a contradiction:

- if $S \in S$, then, by definition of the set S, its element S must have the property that defines this set, i.e., we must have $S \notin S$ – which contradicts to our assumption that $S \in S$;
- on the other hand, if $S \notin S$, then, by definition of the set S, the set S does not have the property that defines this set, i.e., we have $S \in S$ – which contradicts to our assumption that $S \notin S$.

Natural idea. Both paradoxes relate to close loops. The main difference is that the causality paradox appears no matter how long is the loop, while the corresponding paradox of set theory is only known to appear when we consider a on-element loop: $x \in x$. It is therefore reasonable to check whether a similar paradox appears in set theory when we consider loops of arbitrary length.

Main result. In this talk, we show that such an extension is indeed possible, i.e., it is possible to formulate a similar paradox related to a loop $x \in x_1 \in x_2 \in \ldots \in x_n \in x$, for any n. Indeed, let us consider the set $S_n = \{x : \neg\exists x_1, \ldots, x_n : x \in x_1 \in x_2 \in \ldots \in x_n \in x\}$. Then, we have either $S_n \in S_n$ or $S_n \notin S_n$ – and in both cases, we get a contradiction:

- If $S_n \in S_n$, then means that we cannot have sets $x_1, \ldots, x_n$ for which $S_n \in x_1 \in \ldots \in x_n \in S_n$, but we *do* have such sets if we take $x_2 = \ldots = x_n = S_n$ – a contradiction.
- On the other hand, if $S_n \notin S_n$, then there should exist a sequence $x_1, \ldots, x_n$ for which $S_n \in x_1 \in \ldots \in x_n \in S_n$. In particular, this means that $x_n \in S_n$. So, for the element x_n, we have a loop $x_n \in S_n \in x_1 \ldots \in x_{n-1} \in x_n$, which means, by the definition of the set S_n, that x_n cannot be the element of S_n – also a contradiction.