Modal Logic of Closeness in Real and Rational Spaces

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We examine Euclidean spaces equipped with a binary relation that indicates whether two points are closer than a given distance from each other, comparing the geometric properties across differing dimensions of $\mathbb{R}^n$ and $\mathbb{Q}^n$. We are interested in properties definable in the language of modal logic, an efficient formalism for working with relational structures. We found that there are modal formulas which hold in lower dimensions of Euclidean space but do not hold in higher dimensions. Most notably, however, we demonstrate that any modal formula which holds in the rational numbers with the closeness relation also holds in the real numbers, whereas there are modal formulas that are valid in the real number space but not in the rational number space.

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