How to compare situations in which we measure different quantities with different uncertainty

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Formulation of the problem. In many practical situations, we need to design a device for measuring several quantities – e.g., to send on a space mission. Often, this is largely a mission to the unknown – we do not know a priori which measurements will be more important.

In the ideal world, we should measure each of the quantities of interest with maximum accuracy. However, in practice, there are limits on the size and weight of the device, so our options are limited. Under such restrictions, we may have different possible sets of accuracies $a = (a_1, \ldots, a_n)$ for measuring the desired n quantities. Which options should we select?

Let us formulate this problem in precise terms. To select the best option, we need to describe the relation "better of or the same quality" – which we will denote by $a \succeq b$. This relation should be reflexive ($a \succeq a$) and transitive (if $a \succeq b$ and $b \succeq c$, then $a \succeq c$). Of course, if all measurement are more accurate, i.e., if $a_i \le b_i$ for all i, then we should have $a \succeq b$ – and if also $a_i < b_i$ for some i, we should have $b \not\succeq a$.

Since we do not know which quantity is more important, the relation should not change if we swap some quantities. In precise terms, for each permutation π, if $a \succeq b$, then we should have $\pi(a) \succeq \pi(b)$. Finally, the selection should not depend on what measuring unit we use for each quantity: e.g., for measuring length, we can use meters or centimeters. If we switch to a unit which is λ_i times smaller, all numerical values are multiplied by λ_i. Thus, for each tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$, if $(a_1, \ldots, a_n) \succeq (b_1, \ldots, b_n)$, then we should have $(\lambda_1 \cdot a_1, \ldots, \lambda_n \cdot a_n) \succeq (\lambda_1 \cdot b_1, \ldots, \lambda_n \cdot b_n)$. This is known as *scale-invariance.*

Our result: formulation. It turns out that for permutation-invariant and scale-invariant relations, $a \succeq b$ is equivalent to $a_1 \cdot \ldots \cdot a_n \le b_1 \cdot \ldots \cdot b_n$.

Our result: meaning. This result has the following natural interpretation. If we start with some area of values of size $X = X_1 \times \ldots \times X_n$, then we have X_1/a_1 possible different measured values of x_1, etc., with the total of $N = (X_1/a_1) \cdot \ldots \cdot (X_n/a_n)$ combinations. Here, $N = X/(a_1 \cdot \ldots \cdot a_n)$. So, the smaller the product of a_i's, the more alternatives we get and thus, the more information we gain about the studied object.

Proof. For $n = 2$, for all a_1 and a_2, due to permutation-invariance, we have $(\sqrt{a_1}, \sqrt{a_2}) \sim (\sqrt{a_2}, \sqrt{a_1})$, where $a \sim b$ means $a \succeq b$ and $b \succeq a$. For $\lambda_1 = \sqrt{a_1}$ and $\lambda_2 = \sqrt{a_2}$, scale-invariance implies that $(a_1, a_2) \sim (\sqrt{a_1 \cdot a_2}, \sqrt{a_1 \cdot a_2})$. So, by transitivity, if the two options have the same product $a_1 \cdot a_2$, they are equivalent.

For $n > 2$, we can similarly prove that if we replace two values a_i and a_{i+1} with another two values with the same product, the options remain equivalent. Thus, if we start with any option $a = (a_1, \ldots, a_n)$ with $s \stackrel{\rm def}{=} \sqrt[n]{a_1 \cdot \ldots \cdot a_n}$, then we first replace a_1 and a_2 with s and $(a_1 \cdot s_2)/s$. Then, for each k, once we have an equivalent option $(s, \ldots, s, a'_{k+1}, a'_{k+2}, \ldots, a'_n)$, we replace a'_{k+1} and a'_{k+2} with s and $(a'_{k+1} \cdot a'_{k+2})/s$, etc. At the end, we will be able to conclude that the original option a is equivalent to $(s, \ldots, s)$. For such options, the smaller s, the better – and if s is smaller, then s^n is also smaller. Thus, the relative quality of different options is indeed determined by the product s^n of their accuracies: the smaller this product, the better.