How to describe commonsense implication: between conditional probability and logical approach

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How to describe commonsense implication. From the commonsense viewpoint, how can we define the probability P of an implication "if A then B"? From the logical viewpoint, implication is equivalent to $B \vee \neg A$. So, at first glance, it may seem reasonable to define this probability as $P(B \vee \neg A)$. However, this is not always in agreement with common sense: e.g., according to this definition, for $A =$ "contact with a measles person" and $B =$ "has measles", thus defined probability would be 99.999% since most people do not have such a contact. However, intuitively, such a high probability would be wrongly interpreted as a practically guaranteed infection – which, in reality, is not the case.

Another approach is to interpret the desired probability as the conditional probability $P(B \mid A) = P(A \,\&\, B)/P(A)$ – but this is also not always consistent with common sense. Empirical evidence shows that the commonsense probability is between these two values. The following empirical formula was proposed to describe the commonsense value: $P(A \stackrel{\alpha}{\to} B) \stackrel{\text{def}}{=} (P(A \,\&\, B)+\alpha\cdot(1-P(A)))/(P(A)+\alpha\cdot(1-P(A)))$. A natural question is: how to explain this empirical formula?

Analysis of the problem and the resulting explanation. When the implication "if A is B" is absolutely true, then A always implies B. The need to estimate the probability comes from the fact that the implication is not always true. Intuitively, this means that in most (or at least in many) cases, A does imply B, but there are cases when A does not imply B. So it makes sense to focus on these cases, i.e., on the probability $P(\neg B \mid A) = 1 - P(B \mid A)$. This conditional probability has the form $P(A \,\&\, \neg B)/P(A)$.

In general, it is easier to estimate the probability if we have fewer events – e.g., it is usually easier to get statistics about events in El Paso than in the whole state of Texas or in the whole US. Since $A \,\&\, \neg B$ implies A, this means that there more cases when A holds than cases when we have $A \,\&\, \neg B$. Thus, the uncertainty with which we know $P(A \,\&\, \neg B)$ is much smaller than the uncertainty with which we know the probability $P(A)$. In the first approximation, it thus makes sense to ignore the smaller uncertainty and to assume that we know the probability $P(A \,\&\, \neg B)$ exactly – while we know the probability $P(A)$ with uncertainty. This uncertainty means that the actual probability $\overline{P}(A)$ of A may somewhat differ from our estimate $P(A)$. Because of the formula of total probability, we have $\overline{P}(A) = P(A)\cdot P(E)+P(A \mid \neg E)\cdot P(\neg E)$, where E means that our estimate is correct. If we denote $\alpha \stackrel{\text{def}}{=} P(\neg E)$, then we get $\overline{P}(A) = (1-\alpha)\cdot P(A) + \alpha \cdot P(A \mid \neg E)$. We have no information about $P(A \mid \neg E)$, it can be any number from $[0,1]$. So, the smallest value of $\overline{P}(A)$ is $(1-\alpha)\cdot P(A)$, and the largest is $(1-\alpha)\cdot P(A)+\alpha = P(A)+\alpha\cdot(1-P(A))$. Thus, the smallest – guaranteed – value of the ratio $P(\neg B \mid A)$ is $P(A \,\&\, \neg B)/(P(A)+\alpha\cdot(1-P(A)))$. As a result, the largest value of the desired conditional probability $P(B \mid A) = 1 - P(\neg B \mid A)$ is equal to

$$1-\frac{P(A \,\&\, \neg B)}{P(A)+\alpha\cdot(1-P(A))} = \frac{P(A)-P(A \,\&\, \neg B)+\alpha\cdot(1-P(A))}{P(A)+\alpha\cdot(1-P(A))} = \frac{P(A \,\&\, B)+\alpha\cdot(1-P(A))}{P(A)+\alpha\cdot(1-P(A))}.$$

This is exactly the empirical formula that we wanted to explain.