Tutorial on enumeration reducibility and effective mathematics

Mariya I. Soskova

University of Wisconsin–Madison

North American ASL Meeting May 13 – May 16 2025

Outline

- Part 1. Introducing the world of the enumeration degrees: basic definitions and structural properties.
- Part 2. Natural operators and first order definability in the enumeration degrees.
- Part 3. The texture of the enumeration degrees: how effective mathematics gives rise to a zoo of classes.

Part 1. Introducing the world of the enumeration degrees: basic definitions and structural properties.

Motivation

When we define computable functions on $\mathbb{N}$, we naturally include partial functions.

When we define relative computation using Turing reducibility, there is a mismatch: an oracle Turing machine is only well defined for total oracles, but produces partial functions.

Question

How do we extend relative computability so that it makes sense when the oracle is partial?

Attempt 1: Partial reducibility

The following notion was explored by Sasso and Skordev in the 1960-1970s:

Definition

Partial reducibility is the notion we get when we postulate that if during a computation the oracle is queried at an undefined value, the computation does not halt.

Example (Myhill 1961)

If f is partially computable from φ and g is any extension of φ then f is partially computable from g.

Hence this reducibility fails to capture the intuition that if we can enumerate $A \oplus \overline{A}$ then we can compute A. Consider:

$$\varphi(x) = c_{A\oplus\overline{A}}(x) = \begin{cases} 1, & \text{if } x \in A \oplus \overline{A}; \\ \uparrow, & \text{o.w.} \end{cases} \quad \text{and its extension } g(x) = 1.$$

The definition of enumeration reducibility

Myhill suggested introducing non-determinism into the formalization of relative computability.

Definition (Friedberg, Rogers 1959; Uspensky)

$A \leq_e B$ if there is a c.e. set Γ so that

$$x \in A \text{ if and only if } (\exists v)[\langle x, v\rangle \in \Gamma \;\&\; D_v \subseteq B].$$

Myhill's definition of when φ is reducible to ψ is equivalent to $G_\varphi \leq_e G_\psi$.

Every c.e. set can be thought of as inducing an operator on sets: an *enumeration operator*.
We write $A = \Gamma(B)$ to denote that $A \leq_e B$ via Γ.
The elements of an enumeration operator are called *axioms*.

An alternative view of enumeration reducibility

The relation "c.e. in" is not transitive, so it is not a reducibility. Selman explored transitive relations that imply "c.e. in".

Definition (Selman 1971)

AS_1B if and only if for every Turing oracle X we have that if B is c.e. in X then A is c.e. in X.

Note, that $A \leq_e B$ implies AS_1B: if $A = \Gamma(B)$ and X can enumerate B then can X enumerates A via the following procedure:

- Start listing the elements of Γ: $\langle x_1, D_1\rangle$, $\langle x_2, D_2\rangle$, ...
- At the same time start listing B: b_1, b_2, ...
- If you see all elements in D_i listed among b_1, b_2, ... then output x_i.

So there is a uniform procedure for transforming an enumeration of B to an enumeration of A independent of the oracle X.

Selman's theorem

Theorem (Selman 1972, Case 1974)

$A \leq_e B$ if and only if whenever B is X-c.e., A is also X-c.e.

Proof.

Suppose $A \nleq_e B$. We build an enumeration of B that does compute an enumeration of A. Any A-generic enumeration of B will do!

We build $f = \bigcup_s \sigma_s$, where $\sigma_s \in B^{<\omega}$ and $\sigma_s \preceq \sigma_{s+1}$.

At stage s we have σ_s and we ensure that $W_s^f \neq A$.

Find $x \in \omega$ and $\tau \in B^{<\omega}$ such that $x \in W_s^\tau \setminus A$ and $\tau \succeq \sigma_s$ and let $\sigma_{s+1} = \tau$. If no such x and τ exists then $\sigma_{s+1} = \sigma_s$.

If $A = W_s^f$ then $A \leq_e B$ via
$\Gamma = \{\langle x, D\rangle \mid \exists \tau \succeq \sigma_s(x \in W_s^\tau \ \& \ D = \operatorname{ran}(\tau))\}$.

□

The Turing degrees and the enumeration degrees

Definition

Let $\leq$ be either $\leq_T$ or $\leq_e$.

1. $A \equiv B$ if and only if $A \leq B$ and $B \leq A$.
2. $\mathbf{d}(A) = \{B \mid A \equiv B\}$.
3. $\mathbf{d}(A) \leq \mathbf{d}(B)$ if an only if $A \leq B$.
4. Let $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$. Then $\mathbf{d}(A \oplus B) = \mathbf{d}(A) \vee \mathbf{d}(B)$.

And so we have two upper semi-lattices:

1. The Turing degrees $\mathcal{D}_T$ with least element $\mathbf{0}_T$ consisting of all computable sets.
2. The enumeration degrees $\mathcal{D}_e$ with least element $\mathbf{0}_e$ consisting of all c.e. sets.

What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

$A \leq_T B \Leftrightarrow A$ and $\overline{A}$ are c.e. in B $\Leftrightarrow A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(\mathbf{d}_T(A)) = \mathbf{d}_e(A \oplus \overline{A})$, preserves the order, and the least upper bound.

$\mathcal{T} = \iota(\mathcal{D}_T)$ is the set of *total* enumeration degrees. A set A is *total* if $\overline{A} \leq_e A$.

$$(\mathcal{D}_T, \leq_T, \mathbf{0}_T) \cong (\mathcal{T}, \leq_e, \mathbf{0}_e) \subseteq (\mathcal{D}_e, \leq_e, \mathbf{0}_e)$$

On the other hand by Selman's theorem $A \leq_e B$ if and only if the total degrees above $\mathbf{d}_e(A)$ contain the total degrees above $\mathbf{d}_e(B)$.

Non-total degrees exist

The total degrees are small both in terms of measure and category.

Definition

A set X is *quasiminimal* if and only if X is not c.e. and whenever $T \leq_e X$ is total we have that T is computable.

Recall that a set G is *n-generic* if for every Σ^0_n set $W \subseteq 2^{<\omega}$ there is an initial segment of G that is either in W or has no extension in W.

Theorem (Case 1974)

Every 1-generic set is quasiminimal.

A set R is *n-random* if and only if it is not in the intersection of any effectively null sequence of Σ^0_n classes.

Theorem (Cholak, Miller, Soskova)

There are total 1-random sets, but every 2-random set is quasiminimal.

Case's proof

G is *1-generic* if for every Σ^0_1 set $W \subseteq 2^{<\omega}$ there is an initial segment of G that is either in W or has no extension in W.

Theorem (Case 1974)

Every 1-generic set is quasiminimal.

Proof.

Suppose that $\Gamma(G) = T \oplus \overline{T}$ is total.

Let $W = \{\sigma \in 2^{<\omega} \mid (\exists n)[2n, 2n+1 \in \Gamma(\sigma)]\}$.

There must be some $\tau \preceq G$ with no extension in W.

To compute $T(n)$ search for $\sigma \succeq \tau$ so that either $2n$ or $2n+1 \in \Gamma(\sigma)$.

There must be at least one such $\sigma \preceq G$.

If $\sigma_1, \sigma_2 \succeq \tau$ are such that $2n \in \Gamma(\sigma_1)$ and $2n+1 \in \Gamma(\sigma_2)$ then $2n, 2n+1 \in \Gamma(\sigma)$ where $\sigma = \sigma_1 \cup \sigma_2$. □

Properties of the degree structures

Similarities

1. Both $\mathcal{D}_T$ and $\mathcal{D}_e$ are uncountable structures with least element and no greatest element.
2. They have uncountable chains and antichains.
3. They are not lattices: there are pairs of degrees with no greatest lower bound.

Differences

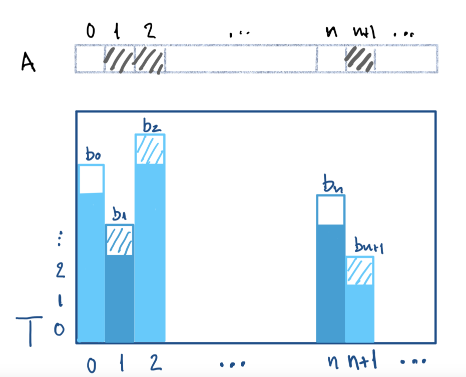
1. (Spector 1956) In $\mathcal{D}_T$ there are *minimal degrees*, nonzero degrees $\mathbf{m}$ such that the interval $(\mathbf{0}_T, \mathbf{m})$ is empty.
2. (Gutteridge 1971) $\mathcal{D}_e$ is downwards dense.

Downwards density of $\mathcal{D}_e$

Theorem (Gutteridge 1971)

There is an enumeration operator Θ so that if $A \not\leq_e \emptyset'$ then $\emptyset <_e \Theta(A) <_e A$.

Proof.



We build a c.e. set $T \subseteq \omega \times \omega$.

The n-th column of T is a finite initial segment of ω ending in b_n.

$\Theta(A)$ is the interior of T plus the b_n where $n \in A$.

So A is computable from $\Theta(A) \oplus \emptyset'$.

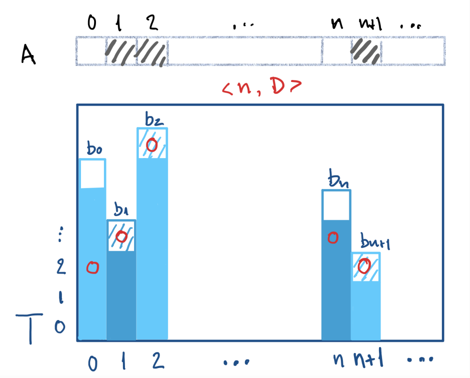
□

Downwards density of $\mathcal{D}_e$

Theorem (Gutteridge 1971)

There is an enumeration operator Θ so that if $A \not\leq_e \emptyset'$ then $\emptyset <_e \Theta(A) <_e A$.

Proof.



We build T so that whether $n \in \Gamma(\Theta(A))$ depends only on the columns that are smaller than n.

So if $A = \Gamma(\Theta(A))$ then A is c.e.

So $\Theta(A) <_e A$.

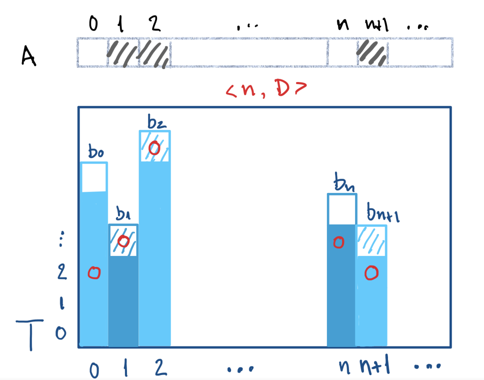
$\square$

Downwards density of $\mathcal{D}_e$

Theorem (Gutteridge 1971)

There is an enumeration operator Θ so that if $A \not\leq_e \emptyset'$ then $\emptyset <_e \Theta(A) <_e A$.

Proof.



We build T so that whether $n \in \Gamma(\Theta(A))$ depends only on the columns that are smaller than n.

So if $A = \Gamma(\Theta(A))$ then A is c.e.

So $\Theta(A) <_e A$.

□

The local structure of the enumeration degrees

Definition

The *local structure* $\mathcal{D}_e(\leq \mathbf{0}_e')$ is the initial interval of all degrees bounded by $\mathbf{0}_e{}' = \mathbf{d}_e(\emptyset')$. These are the enumeration degrees of Σ^0_2 sets.

Theorem (Cooper)

$\mathcal{D}_e(\leq \mathbf{0}_e')$ is dense.

Proof.

A Σ^0_2 set can be approximated by a computable sequence of finite sets $\{A_s\}_{s<\omega}$ so that $n \in A$ if and only if $n \in A_s$ at all but finitely many s and so that there are infinitely many *good* stages s such that $A_s \subseteq A$.

Given $B <_e A$, we build Θ so that $B <_e \Theta(A) \oplus B <_e A$.

At stage s we only add axioms of the form $\langle x, A_s \rangle$ to Θ.

□

Deeper density results

Theorem (Slaman, Sorbi 2014)

There is no initial segment of the enumeration degrees that is linearly ordered.

Definition (Lachlan, Shore 1992)

A set A has a *good approximation* if there is a computable sequence of finite sets $\{A_s\}_{s<\omega}$ so that

1. There are infinitely many good stages s such that $A_s \subseteq A$.
2. For all n the limit over good stages s of $A_s(n)$ exists and is $A(n)$.

Example

Every Σ^0_2 set has a good approximation. So does every set of the form $X \oplus \overline{X}$.

Theorem (Lachlan, Shore 1992)

If $B <_e A$ and A has a good approximation then there is an $B <_e X <_e A$.

Empty intervals of degrees

Theorem (Lachlan, Shore 1992)

There is a Π^0_2 set without a good approximation.

Slaman and Calhoun proved that there are empty intervals of Π^0_2 enumeration degrees.

Theorem (Kent, Lewis-Pye, and Sorbi 2012)

There is a Π^0_2 enumeration degree that is a strong minimal cover.

Here $\mathbf{a}$ is a strong minimal cover of $\mathbf{b}$ if every $\mathbf{x} < \mathbf{a}$ is also below $\mathbf{b}$.

The proof is a sophisticated priority construction using the fact that A is Π^0_2 if and only if it has an approximation $\{A_s\}_{s<\omega}$ so that $n \in A$ if and only if there are infinitely many s so that $n \in A_s$.

Strong embeddings

Definition

Let $\mathcal{L}$ be a finite lattice with least element 0_L and largest element 1_L. We say that $\mathcal{L}$ *strongly embeds* into the enumeration degrees if there is an order preserving function $f \colon \mathcal{L} \to \mathcal{D}_e$ so that if $\mathbf{a} = f(0_L)$ and $\mathbf{b} = f(1_L)$ then the interval $[\mathbf{a}, \mathbf{b}]$ is isomorphic to $\mathcal{L}$ and for any degree $\mathbf{x} < \mathbf{b}$ such that $\mathbf{x}$ is not in the range of f we have that $\mathbf{x} \leq \mathbf{a}$.

Theorem (Lempp, Slaman, Soskova 2021)

Every finite distributive lattice has a strong embedding into the Π^0_2 enumeration degrees.

The existential fragment of the theory of $\mathcal{D}_e$ and $\mathcal{D}_e(\leq \mathbf{0}_e')$

Consider an existential statement in the language of partial orders:

$$(\exists x_1, x_2, \dots x_n)[\dots x_i \leq x_j \wedge x_k \not\leq x_m \dots].$$

To understand the Σ_1-theory of $\mathcal{D}_e$ or $\mathcal{D}_e(\leq \mathbf{0}_e')$ we need to understand what finite partial orders can be embedded into the structure.

We know that $\mathcal{D}_T$ embeds in $\mathcal{D}_e$ and so all finite partial orders do.

Corollary

The Σ_1-theory of $\mathcal{D}_e$ and $\mathcal{D}_e(\leq \mathbf{0}_e')$ is decidable.

Question

What about the Σ_n-theory for $n \geq 2$?

Transfering undecidability

Definition

Let $\mathcal{C}$ be a class of structures in a finite relational language $L = \{R_1, \ldots, R_n\}$. We say that $\mathcal{C}$ is *Σ_k-elementarily definable with parameters* in $\mathcal{D}$ if there are Σ_k-formulas φ_U, φ_{R_i}, and $\varphi_{\neg R_i}$ for $i \leq n$ such that for every $C \in \mathcal{C}$, there are parameters $\vec{\mathbf{p}} \in \mathcal{D}$ that make the structure with universe $U = \{\mathbf{x} | \ \mathcal{D}_e \models \varphi_U(\mathbf{x}, \vec{\mathbf{p}})\}$ and relations defined by φ_{R_i}, $\varphi_{\neg R_i}$ isomorphic to C.

Lemma (Nies 1996)

Let $r \geq 2$ and $k \geq 1$. If a class of models $\mathcal{C}$ is Σ_k-elementarily definable in $\mathcal{D}$ with parameters and the Π_{r+1}-theory of $\mathcal{C}$ is (hereditarily) undecidable, then the Π_{r+k}-theory of $\mathcal{D}$ is (hereditarily) undecidable.

Undecidability of the Π_3-theory

Examples

1. The Π_3-theory of finite distributive lattices in the language of partial orders.
2. The Σ_2-theory of finite bipartite graphs in the language with no equality but one binary edge relation symbol E and unary predicates for the two parts.

Corollary

The Π_3 theory of $\mathcal{D}_e$ is undecidable.

Theorem (Kent 06)

The class of finite bipartite graphs is Σ_1-elementary definable in $\mathcal{D}_e(\leq \mathbf{0}_e')$ and hence the Π_3 theory of $\mathcal{D}_e(\leq \mathbf{0}_e{}')$ is undecidable.

The extension of embeddings problem

There is an algebraic equivalent of deciding which $\forall\exists$-sentences are true in a degree structure $\mathcal{D}$:

Problem

We are given a finite partial order P and finite partial orders $Q_0, \ldots Q_n \supseteq P$. Does every embedding of P in $\mathcal{D}$ extend to an embedding of one of the Q_i?

When $n = 1$ we call this the *extension of embeddings problem*.

In $\mathcal{D}_T$ the extension of embeddings problem is equivalent to the general one.

Theorem (Lerman, Shore 78-88)

Every finite lattice can be embedded as an initial segment of $\mathcal{D}_T$.

If this embedding of P extends to an embedding of some Q_i then every embedding of P does.

The Π_2-theory of $\mathcal{D}_T$ is decidable.

The extension of embeddings $\mathcal{D}_e$ and in $\mathcal{D}_e(\leq \mathbf{0}_e')$

Recall that we can't have initial segment embeddings in the local or global structure of the enumeration degrees.

Theorem (Lempp, Slaman, Soskova 21; Lempp, Slaman, Sorbi 05)

The extension of embeddings problem in $\mathcal{D}_e$ and in $\mathcal{D}_e(\leq \mathbf{0}_e')$ is decidable.

This is not enough:

Example

Consider the partial order P with two elements $a < b$. Consider two extensions adding one more element to P: $Q_1 = \{c < a < b\}$ and $Q_2 = \{a < c < b\}$.

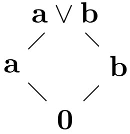
- In $\mathcal{D}_T$ we can embed P as $\mathbf{0}_T < \mathbf{m}$, where $\mathbf{m}$ is minimal, blocking both Q_1 and Q_2.
- In $\mathcal{D}_e$ we can't: if a does not go to $\mathbf{0}_e$ then by downwards density we can extend to Q_1. If it does, we can extend to Q_2.

What's next for $\mathcal{D}_e$

We need to understand what aspects of $\mathcal{D}_e$ determine its Π_2-theory.

Perhaps, a Π_2 sentence φ is true in $\mathcal{D}_e$ if and only if φ is true in every upper semi-lattice U with least element that exhibits end-extensions and downward density.

This implies that there are degrees $\mathbf{a}$ and $\mathbf{b}$ such that: $\mathbf{a}$ and $\mathbf{b}$ are a minimal pair and if $\mathbf{x} < \mathbf{a} \vee \mathbf{b}$ then $\mathbf{x} \leq \mathbf{a}$ or $\mathbf{x} \leq \mathbf{b}$.



Theorem (Jacobsen-Grocott)

If $\mathbf{a}$ and $\mathbf{b}$ are enumeration degrees such that every degree $\mathbf{x} \leq \mathbf{a} \vee \mathbf{b}$ is bounded by $\mathbf{a}$ or bounded by $\mathbf{b}$, then $\{\mathbf{a}, \mathbf{b}\}$ is not a minimal pair.

Open problem. Is the Π_2-theory of $\mathcal{D}_e$ or $\mathcal{D}_e(\leq \mathbf{0}'_e)$ decidable?

Part 2. Natural operators and first order definability in the enumeration degrees.

The search for an extension of the Turing jump

The Turing operator $A \to A'$ is an integral part of degree theory. We want to find the natural extension of the jump operator to non-total enumeration degrees.

Emblematic properties of the Turing jump operator include:

1. It is monotone: $A \leq_T B$ if and only if $A' \leq_1 B'$;
2. It is strictly increasing: $A <_T A'$;
3. Jump inversion: for every $X \geq_T \emptyset'$ there is an A such that $A' \equiv_T X$.

First attempt: the set K_A

Recall that the Turing jump of a set A is $\bigoplus_{e \in \omega} W_e^A = \{\langle e, x \rangle \mid x \in W_e^A\}$, where W_e^A lists all sets that are c.e. in A.

Definition

Let $K_A = \bigoplus_{e \in \omega} \Gamma_e(A)$, where Γ_e lists all enumeration operators.

Consider the properties of $A \to K_A$ with respect to enumeration reducibility.

1. It is monotone: $A \leq_e B$ if and only if $K_A \leq_1 K_B$.
2. It is not strictly increasing: $A \equiv_e K_A$!
3. Jump inversion: trivial.

The skip and the enumeration jump on sets

Definition

The *skip* of A is the set $A^{\Diamond} = \overline{K_A}$.

1. It is monotone: $A \leq_e B$ if and only if $K_A \leq_1 K_B$ iff $\overline{K_A} \leq_1 \overline{K_B}$.
2. It is strict: $A \not\leq_e \overline{K_A}$, but not increasing.

Definition (Cooper 1984)

The *enumeration jump* of A is the set $A' = K_A \oplus \overline{K_A}$.

1. It is monotone: $A \leq_e B$ implies $A' \leq_1 B'$ but this is not always reversible.
2. It is strictly increasing: $A <_e A'$.

The skip and the enumeration jump on degrees

Definition

The enumeration jump of $\mathbf{d}_e(A)$ is $\mathbf{d}_e(A)' = \mathbf{d}_e(K_A \oplus \overline{K_A})$.

The skip of $\mathbf{d}_e(A)$ is $\mathbf{d}_e(A)^\Diamond = \mathbf{d}_e(\overline{K_A})$.

Note that $\mathbf{a}' = \mathbf{a} \vee \mathbf{a}^\Diamond$.

Proposition

If $\mathbf{a} = \iota(\mathbf{x})$ is a total enumeration degree then

$$\mathbf{a}' = \mathbf{a}^\Diamond = \iota(\mathbf{x}').$$

Operator inversion

By definition the range of the enumeration jump consists of total enumeration degrees above $\mathbf{0}_e'$. So where possible, jump inversion for the enumeration jump follows from jump inversion for the Turing jump.

Theorem (McEvoy 1984)

Every total degree $\mathbf{a} \geq \mathbf{0}_e'$ is the jump of a quasiminimal degree.

The skip is not constrained in a similar way!

Theorem (AGLMSS 2021)

Every degree $\mathbf{a} \geq \mathbf{0}_e'$ is the skip of a quasiminimal degree.

Relativization

Soskov proved a very general jump inversion theorem that allows you to control the iterated jumps of a set. A consequence is:

Theorem (Soskov 2000)

For every $\mathbf{x}$ and every total degree $\mathbf{a} \geq \mathbf{x}'$ there is degree $\mathbf{y} \geq \mathbf{x}$ with $\mathbf{y}' = \mathbf{a}$.

Relativization does not work well for the skip :

Theorem (Slaman, Soskova 2025)

For every $\mathbf{x}$ and every degree $\mathbf{a} \geq \mathbf{x}'$ there is degree $\mathbf{y} \geq \mathbf{x}$ with $\mathbf{y}^\Diamond = \mathbf{a}$.

However, there are degrees $\mathbf{x}$ and $\mathbf{a} \geq \mathbf{x}^\Diamond$ so that no degree $\mathbf{y} \geq \mathbf{x}$ has $\mathbf{y}^\Diamond = \mathbf{a}$.

Skip cycles

Recall that $K_A = \bigoplus_e \Gamma_e(A)$.

The set operator that maps A to K_A is monotone: $A \subseteq B$ implies $K_A \subseteq K_B$.

It follows that the double skip operator on sets that maps A to $A^{\Diamond\Diamond} = \overline{K_{\overline{K_A}}}$ has the same property:

$$A \subseteq B \Rightarrow K_A \subseteq K_B \Rightarrow \overline{K_A}^{\Diamond} \supseteq \overline{K_B}^{\Diamond} \Rightarrow \overline{K_{\overline{K_A}}} \subseteq \overline{K_{\overline{K_B}}}$$

The Knaster-Tarski fixed point theorem implies:

Theorem (AGLMSS 2021)

There is a set A so that $A^{\Diamond\Diamond} = A$.

We call such sets *skip 2-cycles*. They are above every hyperarithmetical set.

Note that if A is a skip 2-cycle then A and $A^{\Diamond}$ are incomparable sets that have 1-equivalent jumps: $A' = K_A \oplus \overline{K_A}$ and $(A^{\Diamond})' = \overline{K_A} \oplus \overline{K_{\overline{K_A}}} = \overline{K_A} \oplus A$.

Definability of the enumeration jump

Kalimullin 2003 isolated a definable class of pairs of degrees, later called *$\mathcal{K}$-pairs*.

He showed that $\mathcal{K}$-pairs relativize nicely to any other degree.

He proved that $\mathbf{z}'$ is the largest degree which can be represented as the least upper bound of a triple $\mathbf{a}, \mathbf{b}, \mathbf{c}$, such that each pair $\{\mathbf{a}, \mathbf{b}\}$, $\{\mathbf{b}, \mathbf{c}\}$, and $\{\mathbf{a}, \mathbf{c}\}$ is a $\mathcal{K}$-pair relative to $\mathbf{z}$.

Theorem (Kalimullin 2003)

The enumeration jump is first order definable in $\mathcal{D}_e$.

Question

Is the skip operator first order definable?

$\mathcal{K}$-pairs

Definition (Kalimullin 2003)

A pair of sets $\{A, B\}$ is a *$\mathcal{K}$-pair* if there is a c.e. set W so that:

$$A \times B \subseteq W \text{ and } \overline{A} \times \overline{B} \subseteq \overline{W}.$$

Example

1. If A is c.e. then $\{A, B\}$ is a $\mathcal{K}$-pair for every B as witnessed by $A \times \omega$.
2. If A is a left cut in a computable linear ordering $\leq_L$ on ω then $\{A, \overline{A}\}$ is a $\mathcal{K}$-pair as witnessed by the set $\{\langle n, m\rangle \mid n \leq_L m\}$.

The first kind we call *trivial* and the second *semicomputable*. Jockusch 1968 introduced the semi-computable sets in his thesis.

Properties of $\mathcal{K}$-pairs

Proposition

Let $\{A, B\}$ be a nontrivial $\mathcal{K}$-pair.

1. If $C \leq_e B$ then $\{A, C\}$ is a $\mathcal{K}$-pair.
2. A and B are quasiminimal.
3. $A \leq_e \overline{B}$ and $\overline{A} \leq_e \emptyset' \oplus B$.

Proof.

To see (3) let W witness that $\{A, B\}$ is a $\mathcal{K}$-pair and let
$C = \{a \mid (\exists b)[\langle a, b\rangle \in W \ \& \ b \in \overline{B}]\}$.

$C \subseteq A$ because $\overline{A} \times \overline{B} \subseteq \overline{W}$.

$A \subseteq C$ as otherwise there is some $a^* \in A$ so that $(a^*, b) \in W$ implies $b \in B$ and as $A \times B \subseteq W$ this means $B = \{b \mid \langle a^*, b\rangle \in W\}$ is c.e.

□

Definability of $\mathcal{K}$-pairs

Theorem (Kalimullin 2003)

A pair of sets $\{A, B\}$ is a $\mathcal{K}$-pair if and only if the degrees $\mathbf{d}_e(A) = \mathbf{a}$ and $\mathbf{d}_e(B) = \mathbf{b}$ satisfy

$$(\forall \mathbf{x})[(\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}) = \mathbf{x}.]$$

Proof.

Suppose $\{A, B\}$ is a $\mathcal{K}$-pair witnessed by W and let $Y = \Gamma(A \oplus X) = \Lambda(B \oplus X)$.

$n \in Y$ if and only if there are axioms $\langle n, D_A \oplus D_X \rangle \in \Gamma$ and $\langle n, F_B \oplus F_X \rangle \in \Lambda$ so that $D_A \times F_B \subseteq W$ and $D_X \cup F_X \subset X$.

The reverse direction uses a construction reminiscent of the proof of the Posner-Robinson theorem: we show that if $\{A, B\}$ is not a $\mathcal{K}$-pair there there is total function g so that $A \oplus G_g \geq_e G_{g'}$ and $B \oplus G_g \geq_e G_{g'}$. □

An alternative first order definition of the jump

Theorem (Ganchev, Soskova 2015)

Let $\mathbf{z} > \mathbf{0}_e$. Then $\mathbf{z}'$ is the largest degree that can be represented as the least upper bound of $\mathbf{a} \vee \mathbf{b}$ of a nontrivial $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ with $\mathbf{a} \leq \mathbf{z}$.

Proof.

If $\{A, B\}$ is a $\mathcal{K}$-pair and $A \leq_e Z$ then $B \leq_e \overline{A} \leq_e \overline{K_A} \leq_e \overline{K_Z}$ so $A \oplus B \leq_e Z'$.

For a set X consider $L_X = \{\sigma \in 2^{<\omega} \mid \sigma \leq_{lex} X\}$ and let $R_X = \overline{L_X}$.

$\{L_X, R_X\}$ is a semi-computable $\mathcal{K}$-pair.

$L_X \leq_e X$ and $L_X \oplus R_X \equiv_e X \oplus \overline{X}$.

Consider $\{L_{K_Z}, R_{K_Z}\}$. We have that $L_{K_Z} \leq_e K_Z \equiv_e Z$ and $L_{K_Z} \oplus R_{K_Z} \equiv_e K_Z \oplus \overline{K_Z} = Z'$.

If L_{K_Z} or R_{K_Z} is c.e. then $Z' \leq_e \emptyset'$: we use a priority construction. □

Towards a definition of the total enumeration degrees

Consider this example again: $L_X = \{\sigma \in 2^{<\omega} \mid \sigma \leq_{lex} X\}$ and $R_X = \overline{L_X}$.

$\{L_X, R_X\}$ is a semi-computable $\mathcal{K}$-pair and $L_X \oplus R_X \equiv_e X \oplus \overline{X}$.

Theorem (Jockusch 1968)

A nonzero degree is total if and only if it is the least upper bound of the elements of a non-trivial semi-computable $\mathcal{K}$-pair.

Suppose $\{A, \overline{A}\}$ is a non-trivial $\mathcal{K}$-pair and $A \leq C$.

If $\{C, \overline{A}\}$ is a $\mathcal{K}$-pair then $C \leq_e \overline{\overline{A}} = A$.

Definition (Ganchev, Soskova 2015)

A $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ is *maximal* if whenever $\mathbf{a} \leq \mathbf{c}$, $\mathbf{b} \leq \mathbf{d}$ and $\{\mathbf{c}, \mathbf{d}\}$ is a $\mathcal{K}$-pair we have that $\mathbf{a} = \mathbf{c}$ and $\mathbf{b} = \mathbf{d}$.

Defining totality in $\mathcal{D}_e$

Theorem (Cai, Ganchev, Lempp, Miller, Soskova 2016)

If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair in $\mathcal{D}_e$ then there is a semi-computable set C, such that $A \leq_e C$ and $B \leq_e \overline{C}$.

Proof flavor: Let W be a c.e. set witnessing that a pair of sets $\{A, B\}$ forms a nontrivial $\mathcal{K}$-pair.

1. The countable component: we use W to construct an effective labeling of the computable linear ordering $\mathbb{Q}$.
2. The uncountable component: C will be a left cut in this ordering enumeration reducible to $\overline{B}$.

Theorem (Cai, Ganchev, Lempp, Miller, Soskova 2016)

The set of total enumeration degrees is first order definable in $\mathcal{D}_e$.

The relation *c.e. in*

Recall that ι is the standard embedding of $\mathcal{D}_T$ into $\mathcal{D}_e$.

Theorem (Cai, Ganchev, Lempp, Miller, Soskova 2016)

The set $\{\langle \iota(\mathbf{a}), \iota(\mathbf{x})\rangle \mid \mathbf{a} \text{ is c.e. in } \mathbf{x}\}$ is first order definable in $\mathcal{D}_e$.

Proof.

A Turing degree $\mathbf{a}$ is c.e. in a nonzero Turing degree $\mathbf{x}$ if and only if $\iota(\mathbf{a})$ is the least upper bound of a maximal $\mathcal{K}$-pair with one side bounded by $\iota(\mathbf{x})$.

A result by Cai and Shore implies that $\mathbf{a}$ is c.e. iff $\mathbf{a} \vee \mathbf{b}$ is c.e. in $\mathbf{b}$ for every $\mathbf{b} \not\leq_T \mathbf{0}'_T$. □

Consequences: the full theory

Theorem (Simpson 77)

The theory of $\mathcal{D}_T$ is computably isomorphic to second order arithmetic.

The fact that $\leq_e$ is arithmetically definable and the definability of the total degrees yield:

Corollary (Slaman, Woodin 1997)

The theory of $\mathcal{D}_e$ is computably isomorphic to second order arithmetic.

The automorphism question

A long standing open questions in degree theory is whether $\mathcal{D}_T$ has a nontrivial automorphism.

Definition

A set B is an *automorphism base* for a structure $\mathcal{A}$ if whenever π_1 and π_2 are automorphisms of $\mathcal{A}$ that agree on B they are the same: $\pi_1 = \pi_2$.

Theorem (Slaman, Woodin 1986)

The Turing degrees have at most countably many automorphisms. There is a single Turing degree $\mathbf{g} \leq \mathbf{0}^{(5)}$ that forms an automorphism base.

The total degrees as an automorphism base

By Selman's theorem $\mathbf{d}_e(A)$ is determined by the total degrees above it.

Corollary

The total enumeration degrees form a definable automorphism base of the enumeration degrees.

- A nontrivial automorphism of $\mathcal{D}_e$ induces a nontrivial automorphism of $\mathcal{D}_T$.
- $\mathcal{D}_e$ has at most countable many automorphisms.
- The total degrees below $\mathbf{0}_e^{(5)}$ are an automorphism base of $\mathcal{D}_e$.

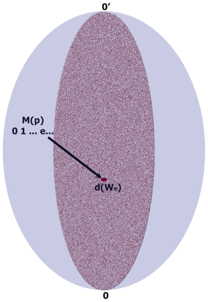
Theorem (Slaman, Soskova 2016)

The total degrees below $\mathbf{0}_e'$ are an automorphism base of $\mathcal{D}_e$, and so a nontrivial automorphism of $\mathcal{D}_e$ induces a nontrivial automorphism of $\mathcal{D}_e(\leq \mathbf{0}_e')$ and $\mathcal{D}_T$.

An indexing of the c.e. degrees

Definition

We say that $\vec{\mathbf{p}}$ *code* a structure $\mathcal{A} = (A, R_1, \dots R_k)$ in a degree structure $\mathcal{D}$ if there are formulas φ_U, φ_{R_i} that make the structure $\mathcal{M}$ with universe $U = \{\mathbf{x} | \mathcal{D}_e \models \varphi_U(\mathbf{x}, \vec{\mathbf{p}})\}$ and relations defined by φ_{R_i} isomorphic to $\mathcal{A}$.



Theorem (Slaman, Woodin 1990)

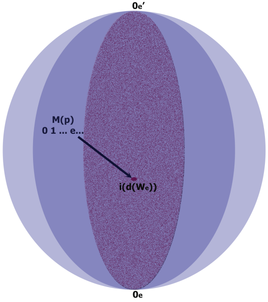
There are parameters $\vec{\mathbf{p}}$ computable from $\mathbf{0}'$ which code the standard model of first order arithmetic together with a function $\psi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_T$ such that $\psi(e^{\mathcal{M}}) = d_T(W_e)$.

We call ψ *an indexing* of the c.e. Turing degrees.

An indexing of the Π^0_1 degrees

Theorem (Slaman, Woodin 1990)

There are total parameters $\vec{\mathbf{p}}$ below $\mathbf{0}'_e$ which code the standard model of first order arithmetic together with a function $\psi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_T$ such that $\psi(e^{\mathcal{M}}) = d_e(W_e \oplus \overline{W_e})$.



If π_1 and π_2 are automorphisms that agree on $\vec{\mathbf{p}}$ then they agree on every Π^0_1 enumeration degree.

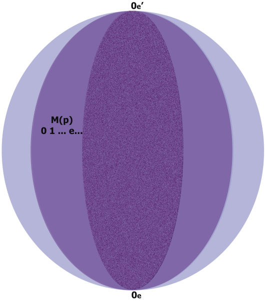
The plan is to extend the indexing ψ so that it determines all degrees below $\mathbf{0}^{(5)}_e$. We already know that this is an automorphism base.

Capturing all total degrees below $\mathbf{0}_e{}'$

A degree $\mathbf{a}$ is low if $\mathbf{a}' = \mathbf{0}_e'$.

We prove that we can identify every distinct low enumeration degree by its position relative the Π^0_1 enumeration degrees.

This allows us to extend ψ so that it indexes all low enumeration degrees.



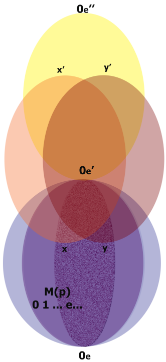
Every total degree below $\mathbf{0}_e'$ is the join of two low enumeration degrees: a maximal $\mathcal{K}$-pair below $\mathbf{0}_e'$ is low.

This allows us to extend ψ so that it indexes all total degrees below $\mathbf{0}_e'$.

Moving outside the local structure

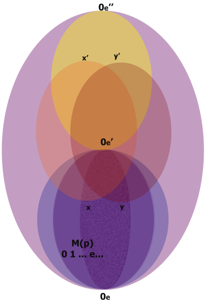
We next extend our previous result to capture all total degrees that are "c.e. in" and above some total Δ^0_2 enumeration degree.

- Sacks jump inversion and the fact that the jump is definable for degrees above $\mathbf{0}'_e$.
- Sacks splitting theorem, a new priority construction and the fact that the image of the relation "c.e. in" is definable for degrees incomparable to $\mathbf{0}'_e$.



Relativizing the previous step we get all total degrees in an interval of the form $[\mathbf{x}, \mathbf{x}']$, where $\mathbf{x} \leq \mathbf{0}'_e$ is total.

Iterating



We finally use a forcing construction and properties of 2-generic sets to show that we can further capture all total degree below $\mathbf{0}_e''$.

So from an indexing of all total degrees below $\mathbf{0}_e'$ we get an indexing of all degrees below $\mathbf{0}_e''$.

Iterating this result we extend our indexing to capture all total degrees below $\mathbf{0}_e^{(5)}$.

But that, we know already, is an automorphism base.

Question

Are there nontrivial automorphisms of $\mathcal{D}_e(\leq \mathbf{0}_e')$?

Part 3. The texture of the enumeration degrees: how effective mathematics gives rise to a zoo of classes.

Enumeration reducibility in computable structure theory

Let $\mathcal{A}$ be a countable structure. If $\mathcal{B}$ is isomorphic to $\mathcal{A}$ and has domain ω, we say that $\mathcal{B}$ is a *copy* of $\mathcal{A}$. We identify $\mathcal{B}$ with its atomic diagram.

Definition (Jockusch, Richter 1981)

The *degree spectrum* of $\mathcal{A}$ is the set of Turing degrees of copies of $\mathcal{A}$.

We say that $\mathcal{A}$ has *Turing degree* $\mathbf{x}$ if $\mathbf{x}$ computes a copy of $\mathcal{A}$ and every copy of $\mathcal{A}$ computes $\mathbf{x}$.

If $\mathcal{A}$ has Turing degree $\mathbf{x}$ then its degree spectrum is the Turing cone above $\mathbf{x}$.

Not every structure has a Turing degree.

The enumeration degree of a structure

Definition

We say that $\mathcal{A}$ has *enumeration degree* $\mathbf{d}_e(X)$ if every enumeration of X computes a copy of $\mathcal{A}$ and every copy of $\mathcal{A}$ computes an enumeration of X.

If $\mathcal{A}$ has enumeration degree $\mathbf{x}$ then its degree spectrum maps to the set of total degrees above $\mathbf{x}$—the *enumeration cone* above X.

If $\mathcal{A}$ has enumeration degree then that degree is the enumeration degree of some $\exists$-type of some tuple in $\mathcal{A}$.

If $\mathcal{A}$ has Turing degree $\mathbf{d}_T(A)$ then $\mathcal{A}$ has enumeration degree $\mathbf{d}_e(A \oplus \overline{A})$.

Examples of structures with enumeration degree

Each of the following classes of structures always have enumeration degree and every enumeration degree is the degree of some such structure.

1. (Calvert, Harizanov, Shlapentokh 07) Torsion-free abelian groups of finite rank;
2. (Frolov, Kalimullin, Miller 09) Fields of finite transcendence degree over $\mathbb{Q}$.
3. (Steiner 13) Graphs of finite valence with finitely many connected components.

Note! If the enumeration degree of a structure $\mathcal{A}$ is non-total then $\mathcal{A}$ does not have Turing degree.

Computable metric spaces

Definition

A *computable metric space* is a metric space $\mathcal{M}$ together with a countable dense sequence $Q^{\mathcal{M}} = \{q_n^{\mathcal{M}}\}_{n\in\omega}$ on which the metric is computable (as a function $\omega^2 \to \mathbb{R}$).

Example

The *Hilbert cube* is $[0,1]^\omega$ with the metric

$$d(\alpha,\beta) = \sum_{n\in\omega} |\alpha(n) - \beta(n)|/2^n.$$

Let $Q^{[0,1]^\omega}$ be the sequences of rationals in $[0,1]$ with finite support.

Other computable metric spaces include 2^ω, ω^ω, $\mathbb{R}$, and $\mathcal{C}[0,1]$.

Points in computable metric spaces

Definition

$\lambda\colon \mathcal{Q}^+ \to \omega$ is a *name* of a point $x \in \mathcal{M}$ if for all rationals $\varepsilon > 0$ we have $d_{\mathcal{M}}(x, q^{\mathcal{M}}_{\lambda(\varepsilon)}) < \varepsilon$.

As before, the complexity of a point in a metric space can be captured through the collection of Turing degrees of names of this point.

Example

Fix a real number $r \in \mathbb{R}$ with the usual metric. Let $D_r = \{q \in \mathbb{Q} \mid q < r\}$.

Every name λ for r can compute D_r: If r is not rational then $q < r$ if and only if there is some small ε so that $\lambda(\varepsilon) - q > \varepsilon$ and $q > r$ if and only if there is some small ε so that $q - \lambda(\varepsilon) > \varepsilon$. If r is rational then D_r is computable.

D_r can compute a name λ_r for r, a name of least Turing degree.

Degrees of points in computable metric spaces

Question (Pour El and Lempp)

Do elements of computable metric spaces have least Turing degree names?

Definition (Miller 2004)

If x and y are members of (possibly different) computable metric spaces, then $x \leq_r y$ if there is a uniform way to compute a name for x from a name for y.

This reducibility induces the *continuous degrees*.

Theorem (Miller 2004)

Every continuous degree contains a point from $[0,1]^\omega$ and a point from $C[0,1]$.

Embedding the continuous degrees into the e-degrees

For $\alpha \in [0,1]^\omega$, let

$$C_\alpha = \bigoplus_{i\in\omega} \{q \in \mathbb{Q} \mid q < \alpha(i)\} \oplus \{q \in \mathbb{Q} \mid q > \alpha(i)\}.$$

Observation. Enumerating C_α is exactly as hard as computing a name for α. So $\alpha \mapsto C_\alpha$ induces an embedding of the continuous degrees into the enumeration degrees.

- Elements of 2^ω, ω^ω, and $\mathbb{R}$ are mapped onto the *total* degree of their least Turing degree name (i.e., their Turing degree).
- A point $x \in \mathcal{M}$ has nontotal (enumeration) degree iff it has no least Turing degree name.

So Pour El and Lempp's question becomes: are there nontotal continuous degrees.

Nontotal continuous degrees

Theorem (Miller 2004)

There is a nontotal continuous degree.

Proof.

- If $x \in [0,1]^\omega$ has total degree, then there is a $y \in 2^\omega$ and Turing functionals Γ, Ψ that map (names of) x to (names of) y and back.
- The subspaces on which the functions induced by Γ and Ψ are inverses are homeomorphic (because computable functionals induce continuous functions).
- Subspaces of 2^ω are zero dimensional, so if $x \in [0,1]^\omega$ has total degree, then it is in one of *countably many* zero dimensional "patches".
- The Hilbert cube $[0,1]^\omega$ is strongly infinite dimensional, hence not a countable union of zero dimensional subspaces.
- So some $x \in [0,1]^\omega$ is not covered by one of these patches. $\square$

Other proofs invoke Sperner's lemma or a variant of Brouwer's fixed point theorem to multivalued functions on an infinite dimensional space.

The structure of nontotal continuous degrees

Definition

A Turing degree $\mathbf{a}$ is *PA*

- if $\mathbf{a}$ computes a complete extension of Peano Arithmetic, or equivalently
- if $\mathbf{a}$ computes a path in every infinite computable tree.

The degree $\mathbf{a}$ is *PA above* $\mathbf{b}$ if $\mathbf{a}$ computes a path in every infinite $\mathbf{b}$-computable tree.

Theorem (Miller 2004)

The total degrees below a nontotal continuous enumeration degree form a *Scott set*, an ideal closed under the relation "PA above".

The total (Turing) degree $\mathbf{a}$ is *PA above* the total (Turing) degree $\mathbf{b}$ if and only if there is a nontotal continuous degree $\mathbf{c}$ such that $\mathbf{b} < \mathbf{c} < \mathbf{a}$.

Almost total degrees

As it turns out, the continuous enumeration degrees have a very simple characterization inside the enumeration degrees.

Definition

An enumeration degree $\mathbf{a}$ is *almost total* if whenever $\mathbf{b} \not\leq \mathbf{a}$ is total, $\mathbf{a} \vee \mathbf{b}$ is also total.

Theorem (Andrews, Igusa, Miller, Soskova)

Almost total degrees are continuous.

Proof.

The proof is in several steps and uses an application of the effective version of the Urysohn Metrization Theorem proved by Schröder (1998). □

Definability of the continuous degrees

Corollary

The continuous degrees are first order definable in the enumeration degrees.

The relation “PA above” between total degrees is first order definable in the enumeration degrees.

All known constructions of nontotal continuous degrees involve a nontrivial topological component.

Conversely, Kihara and Pauly proved that the fact that the Hilbert cube is not a countable union of subspaces of Cantor space follows from the fact that there is a nontotal continuous degrees in every cone.

So a purely topological fact is reflected in the structure of the enumeration degrees.

PA relative to an enumeration oracle

You can extend relations defined on Turing degrees to enumeration oracles by replacing "c.e. in" with "$\leq_e$":

Definition

A set $U \subseteq 2^\omega$ a $\Sigma^0_1\langle A\rangle$ *class* if there is a set of strings $W \leq_e A$, such that

$$U = [W] = \{X \in 2^\omega \mid (\exists \sigma \in W)\ X \succeq \sigma\}.$$

A $\Pi^0_1\langle A\rangle$ *class* is the complement of a $\Sigma^0_1\langle A\rangle$ class.

$\langle B\rangle$ is PA relative to $\langle A\rangle$ if B enumerates (the initial segments of) a member of every nonempty $\Pi^0_1\langle A\rangle$ class.

Thus, B is PA above A if and only if $\langle B \oplus \overline{B}\rangle$ is PA above $\langle A \oplus \overline{A}\rangle$.

Lifting properties from the Turing world

Transferring theorems from the Turing degrees to total oracles we have:

1. No total degree $\mathbf{a}$ is *$\langle$self$\rangle$-PA*: $\mathbf{a}$ is not PA relative to $\mathbf{a}$.
2. If $\mathbf{b}$ is PA relative to $\mathbf{a}$ and $\mathbf{a}$ is total then $\mathbf{b} \geq \mathbf{a}$. We say that $\mathbf{a}$ is *PA-bounded*.
3. If $\mathbf{a}$ is total then it has a *universal class*: a $\Pi^0_1\langle\mathbf{a}\rangle$ class P such that $\langle\mathbf{b}\rangle$ is PA relative to $\langle\mathbf{a}\rangle$ if and only if $\mathbf{b}$ enumerates a member of P.

Theorem (Franklin, Lempp, Miller, Schweber, and Soskova 2019)

The continuous degrees are exactly the PA-bounded degrees. They cannot be $\langle$self$\rangle$-PA and they have universal classes.

Theorem (Miller, Soskova 2014)

There are $\langle$self$\rangle$-PA oracles. They cannot have universal oracles.

Investigating the oracles with a universal class lead to the introduction and exploration of many other classes.

Symbolic dynamics

The *shift operator* on 2^ω is the map taking an infinite binary sequence $\alpha \in 2^\omega$ to the unique $\beta \in 2^\omega$ such that $\alpha = a\beta$ for some $a \in \{0, 1\}$, i.e., the operator that erases the first bit of the sequence.

Definition

- A *subshift* is closed, shift-invariant subspace X of 2^ω.
- The *degree spectrum* of a subshift X is the set $Spec(X)$ of Turing degrees of elements of the subshift.
- X is a *minimal subshift* if no nonempty $Y \subset X$ is a subshift.

If $Spec(X)$ has a least element, then it could be considered as the *Turing degree* of the subshift X.

The spectrum of a minimal subshift

Given a minimal subshift X, we would like to characterize the set of Turing degrees of members of X.

Definition

The *language* of subshift $X \subseteq 2^\omega$ is the set

$$L_X = \left\{\sigma \in 2^{<\omega} \mid (\exists \alpha \in X)\ \sigma \text{ is a subword of } \alpha\right\}.$$

1. If X is minimal and $\sigma \in L_X$, then for every $\alpha \in X$, σ is a subword of α. So every element of X can enumerate the set L_X.
2. If we can enumerate L_X, then we can compute a member of X.

Theorem (Jeandel 2015)

A Turing degree $\mathbf{a}$ computes a member of the minimal subshift X if and only if $\mathbf{a}$ can enumerate L_X.

A special property of the language of a minimal subshift

Jeandel noticed something special about L_X for a minimal subshift X.

- An enumeration of $\overline{L_X}$ allows us to eliminate branches that do not belong to X in a stage by stage manner.
- If w is word that appears along every branch that remains at stage s, then $w \in L_X$.
- The compactness of 2^ω ensures that we won't miss any word from the language using this process of enumeration.

So $L_X \leq_e \overline{L_X}$.

Cotototal sets and degrees

Definition

A set A is *cototal* if $A \leq_e \overline{A}$. An enumeration degree is *cototal* if it contains a cototal set.

Example

1. Every total enumeration degree is cototal: $\overline{A \oplus \overline{A}} \equiv_e A \oplus \overline{A}$.
2. Every Σ^0_2 set A has cototal degree because $A \equiv_e K_A \leq \emptyset'_e \leq_e \overline{K_A}$.
3. Every continuous degree is cototal: $C_\alpha \leq_e \overline{C_\alpha}$ because $q < \alpha$ if and only if there is some rational r so that $q < r \leq \alpha$.

Characterizations of the cototal enumeration degrees

Theorem (McCarthy 2018)

Every cototal enumeration degree is the degree of the language of a minimal subshift.

Theorem (McCarthy 2018)

The cototal enumeration degrees are the degrees of complements of maximal antichains in $\omega^{<\omega}$.

Theorem (Montalban 2015, McCarthy 2018)

If the degree spectrum of a structure, viewed as a subset of Cantor space, is F_σ then it is the enumeration cone of a cototal degree.

The cototal enumeration degrees as a substructure of the enumeration degrees

Recall that sets with *good approximations* behave nicely in terms of density:

Theorem (Harris; Miller, Soskova 2018)

The good enumeration degrees are exactly the cototal enumeration degrees.

Theorem (Miller, Soskova 2018)

The cototal enumeration degrees are dense.

Theorem (AGKLMSS 2019)

A degree $\mathbf{a}$ is cototal if and only if $\mathbf{a}' = \mathbf{a}^{\Diamond}$.

Topological classification of classes of e-degrees

Definition (Kihara, Pauly 2018)

A *represented space* is a pair of a second countable T_0 topological space X and listing of an open basis $B^X = \{B_i\}_{i<\omega}$.

A name for a point $x \in X$ is an enumeration of the set $N_x = \{i \mid x \in B_i\}$.

Thus a represented space X gives rise to a class of e-degrees $\mathcal{D}_X \subseteq \mathcal{D}_e$.

Example (Kihara, Ng, and Pauly 2019)

- $\mathcal{D}_{S^\omega} = \mathcal{D}_e$, where S is the Sierpinski topology $\{\emptyset, \{1\}, \{0,1\}\}$.
- $\mathcal{D}_{2^\omega} = \mathcal{D}_{\mathbb{R}}$ is the total enumeration degrees.
- $\mathcal{D}_{[0,1]^\omega}$ is the continuous degrees.
- $\mathcal{D}_{\mathbb{R}_{\frac{1}{2}}}$ is the semicomputable degrees.
- The effectively G_δ spaces give rise to the cototal degrees.

The graph cotototal degrees

Question

Can you cover $[0,1]^\omega$ with countably many homeomorphic copies of subspaces of $\mathbb{N}^\omega_{cof}$?

Definition

A degree is *graph cotototal* if it contains $\overline{G_f}$ for a total f.

Theorem (AGKLMSS 2019)

There are cotototal degrees that are not graph-cotototal.

Theorem (Kihara, Ng, and Pauly 2019)

The graph-cotototal degrees are the degrees in $\mathcal{D}_{\mathbb{N}^\omega_{cof}}$.

Question

Is there a continuous degree that is not graphcotototal?

The end

- $\mathcal{D}_e$ is a degree structure with intricate connections to $\mathcal{D}_T$ and many interesting open questions.
- Looking through the lens of effective mathematics gives us a rich variety of classes of enumeration degrees.
- Which of them are definable?