Verified Collaboration:
How Lean is Transforming Mathematics, Programming, and AI

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Chief Architect, Lean FRO

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Breaking the Cycle of Uncertainty: Math, Software, and AI You Can Trust

Math, software, and AI often rely on **manual review** or **partial testing**.

An error in a theorem or critical software system can have massive consequences.

Progress dies where fear of mistakes lives.

Breaking the Cycle of Uncertainty: Math, Software, and AI You Can Trust

Math, software, and AI often rely on **manual review** or **partial testing**.

An error in a theorem or critical software system can have massive consequences.

Progress dies where fear of mistakes lives.

Lean: **machine-checkable proofs eliminate guesswork and create trust**.

If every step is formally verified, we unlock unprecedented confidence and collaboration.

Lean is an open-source programming language and proof assistant that is transforming how we approach mathematics, software verification, and AI.

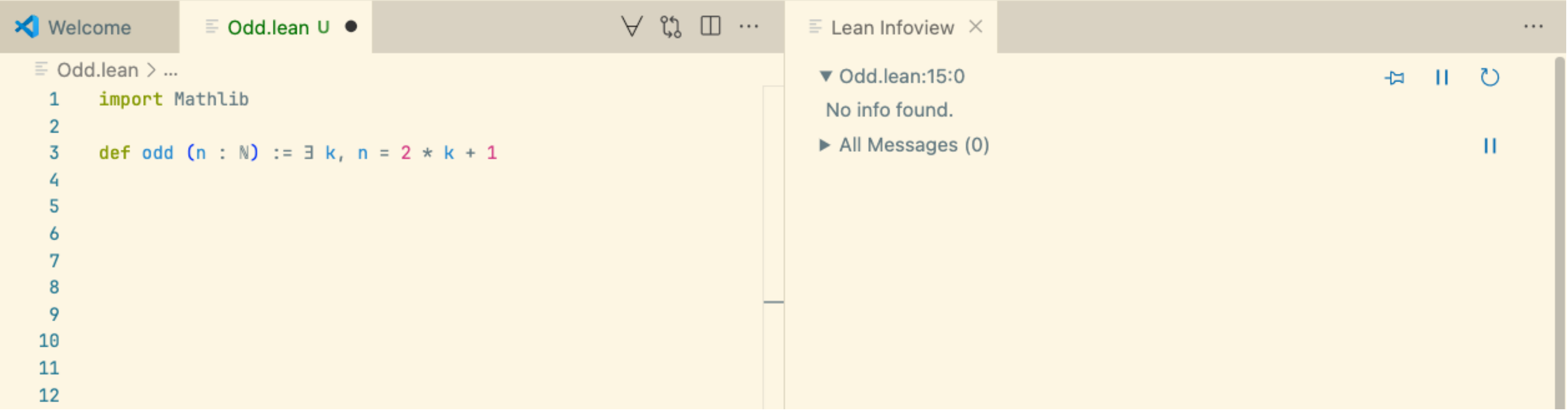
The Lean project, started in 2013, aimed at merging interactive and automated theorem proving.

Lean provides **machine-checkable proofs**.

Lean addresses the “trust bottleneck”.

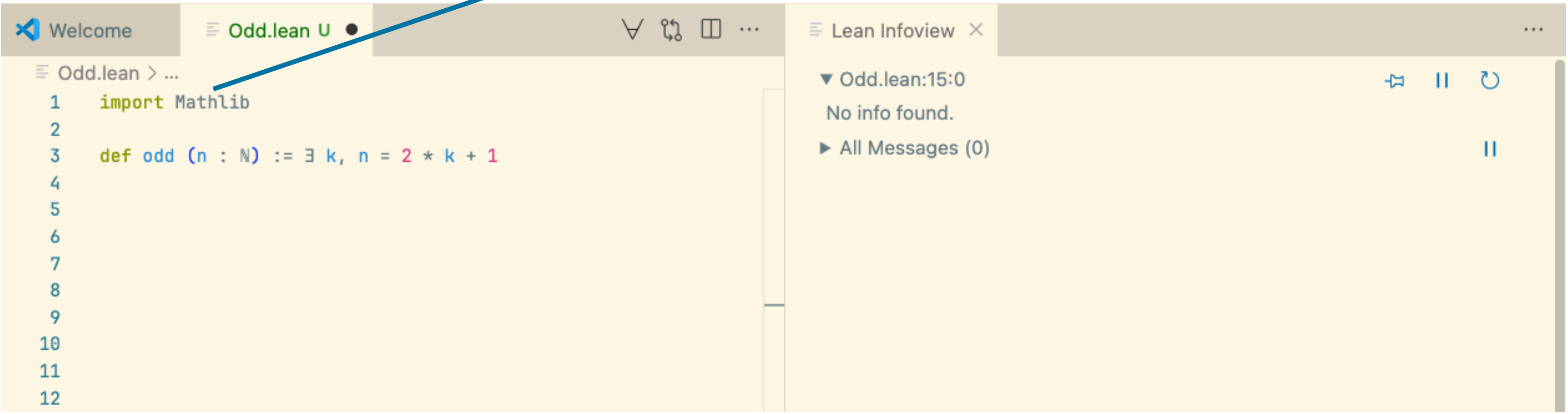
Lean opens up new possibilities for collaboration.

A small example

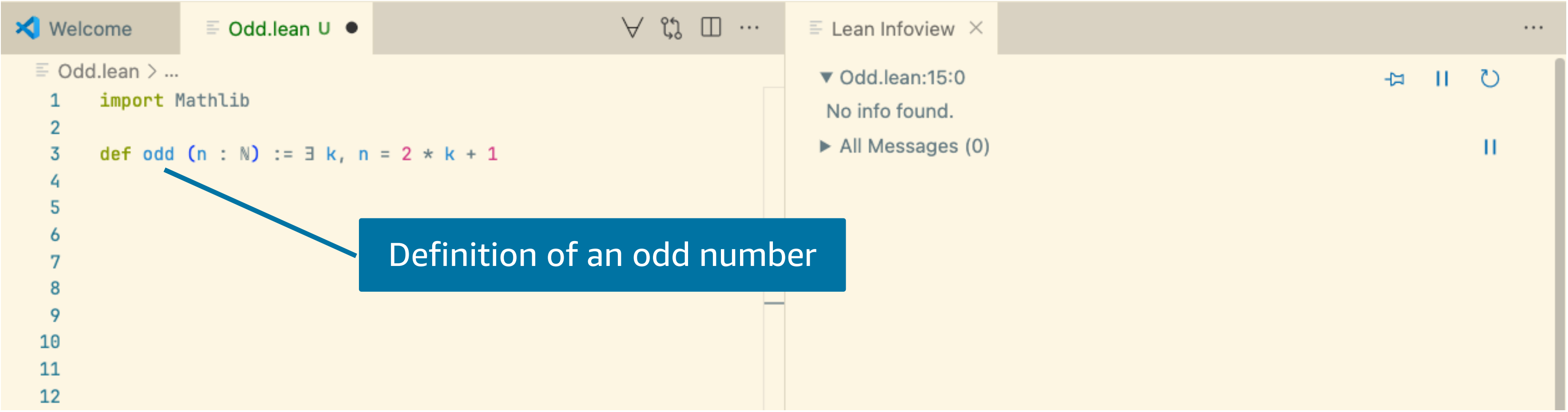


A small example

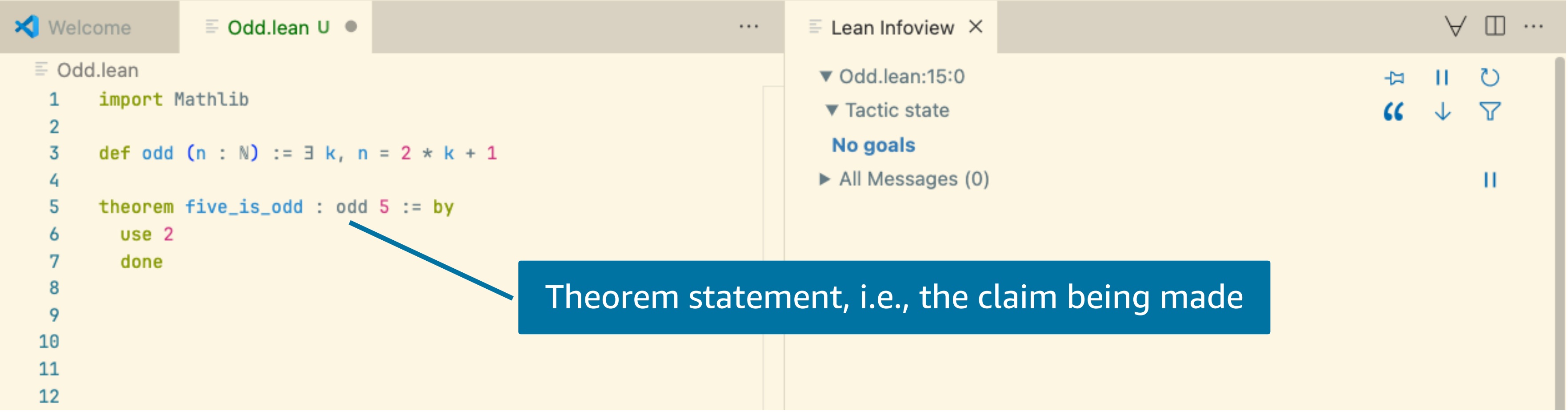
Mathlib is the Lean Mathematical library



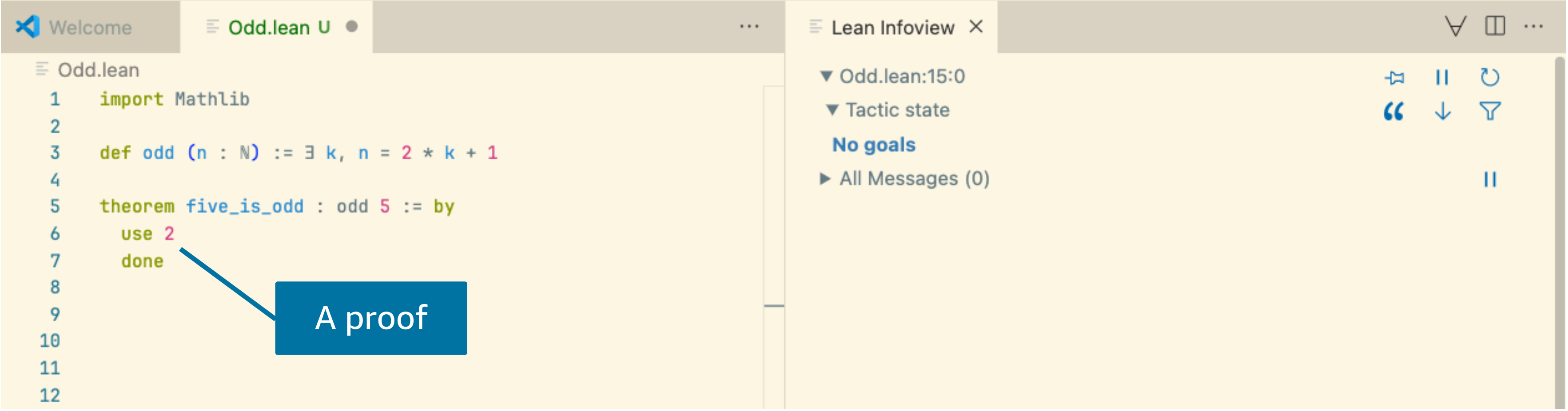
A small example



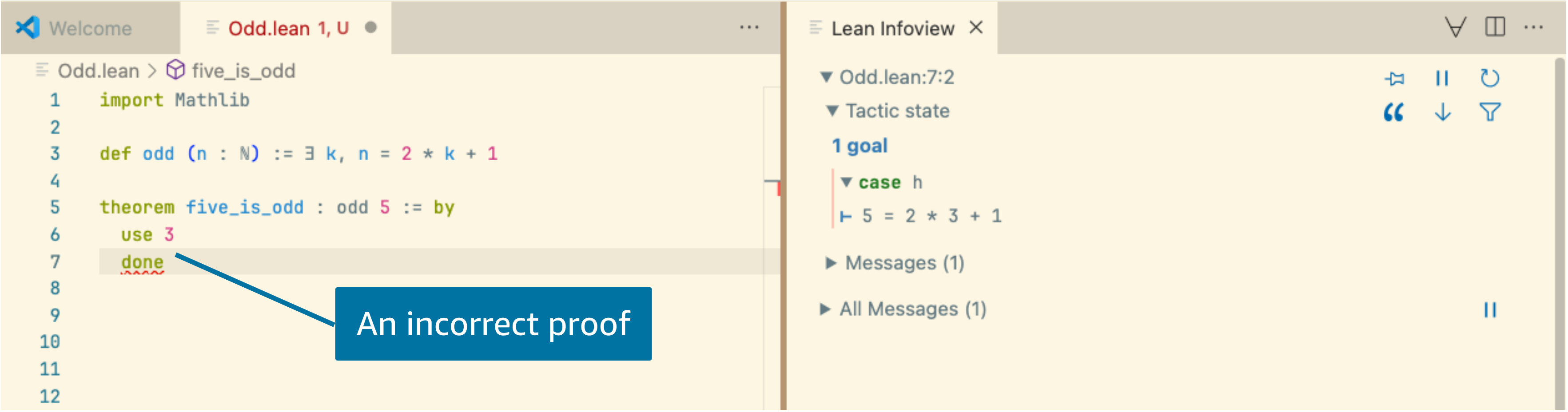
Our first theorem



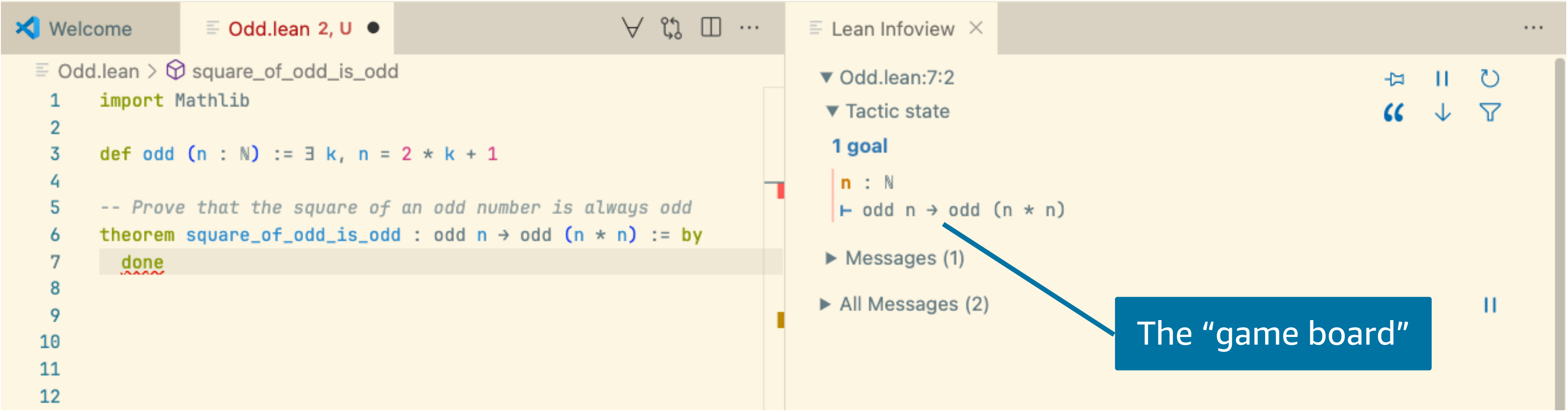
Our first theorem



Our first theorem

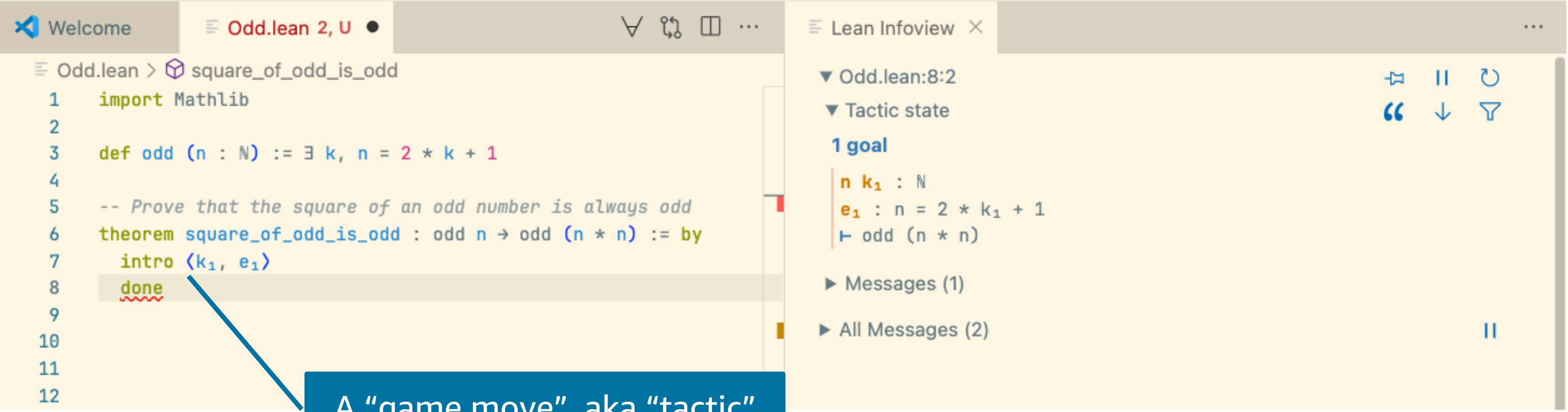


Theorem proving in Lean is an interactive game



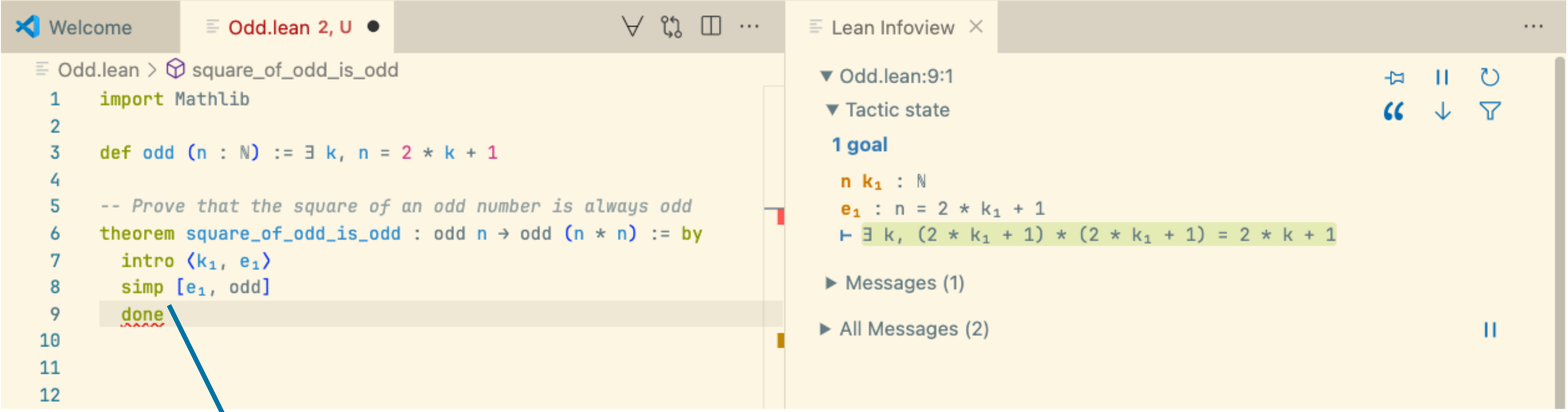
"You have written my favorite computer game", Kevin Buzzard

Theorem proving in Lean is an interactive game



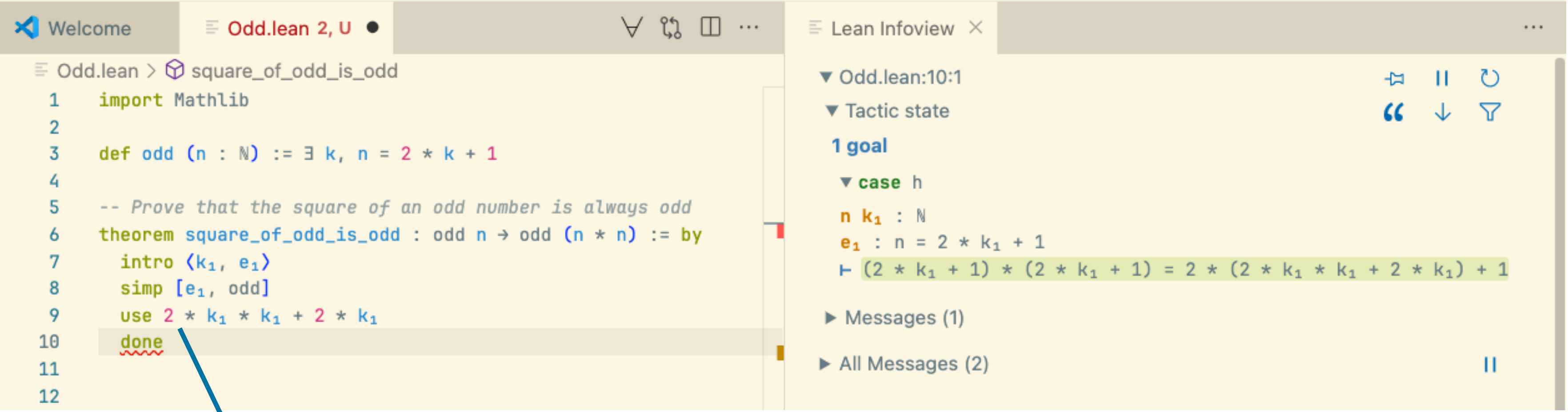
A “game move”, aka “tactic”

Theorem proving in Lean is an interactive game



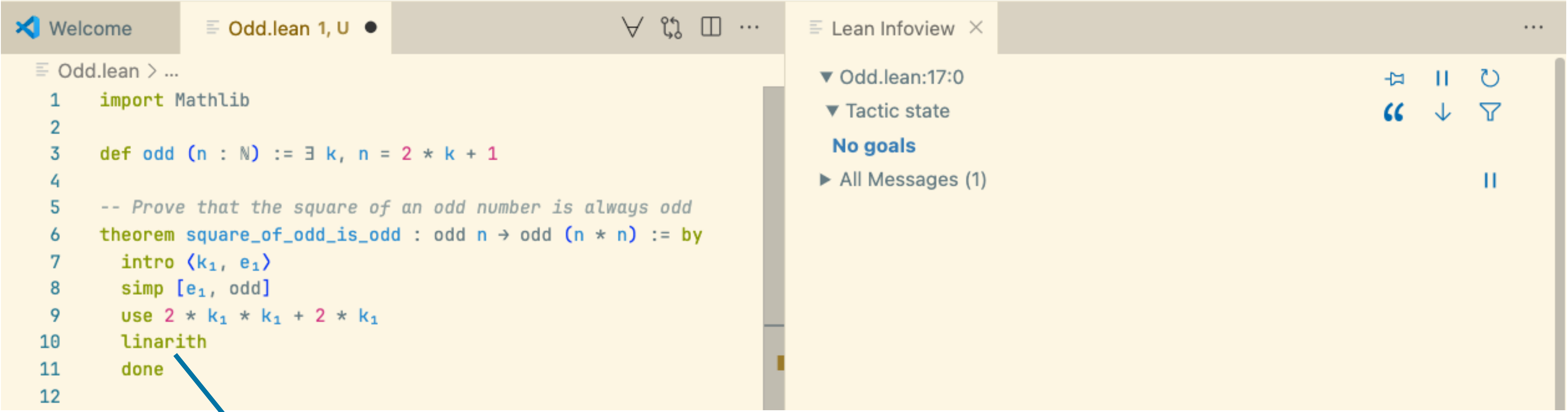
The "game move" `simp`, the simplifier, is one of the most popular moves in our game

Theorem proving in Lean is an interactive game



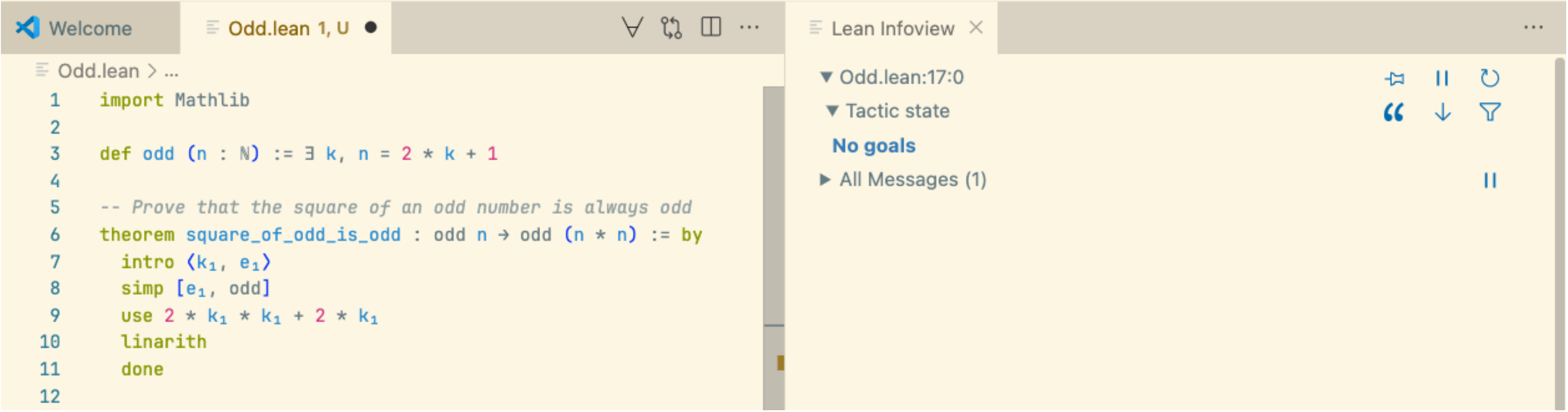
The “game move” `use` is the standard way of proving statements about existentials

Theorem proving in Lean is an interactive game



We complete this level using `linarith`, the linear arithmetic, move

Theorem proving in Lean is an interactive and addictive game



"You can do 14 hours a day in it and not get tired and feel kind of high the whole day.

You're constantly getting positive reinforcement", Amelia Livingston

Mathlib

The Lean Mathematical Library supports a wide range of projects.

It is an open-source **collaborative project** with over 500 contributors and 1.8M LoC.

“I’m investing time now so that somebody in the future can have that amazing experience”,

Heather Macbeth

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FOUNDATIONS OF MATHEMATICS

Building the Mathematical Library of the Future

Mathematics

Software

AI

Mathematics

Preamble: the Perfectoid Spaces Project

Kevin Buzzard, Patrick Massot, Johan Commelin

Goal: Demonstrate that we can **define complex mathematical objects** in Lean.

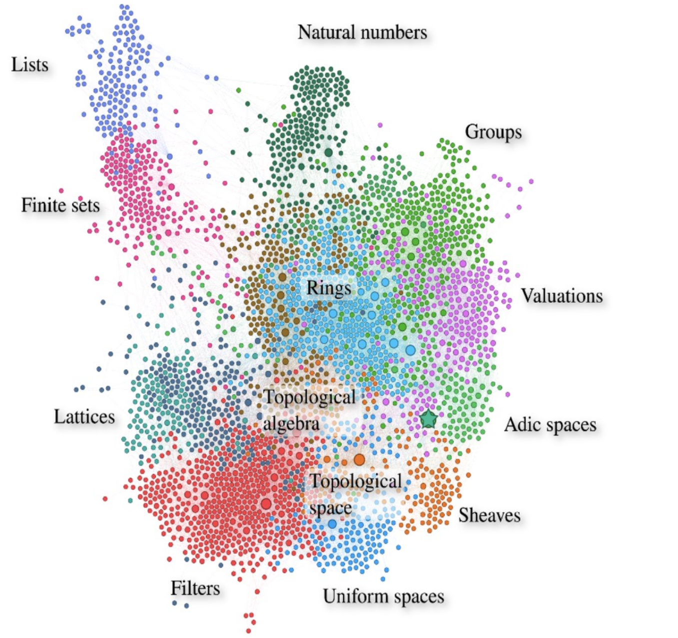
They translated Peter Scholze's definition into a form a computer can understand.

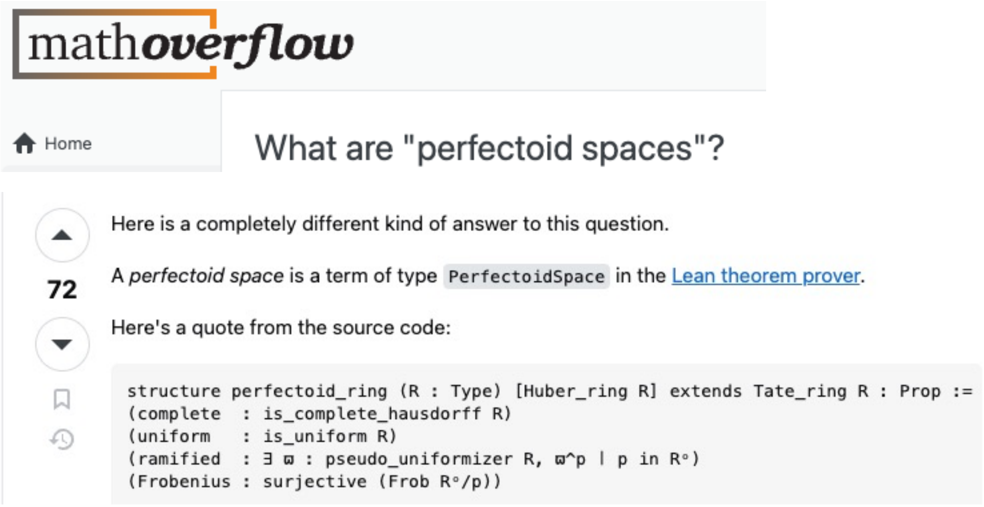
It not only achieved its goals but also demonstrated to the math community that **formal objects can be visualized and inspected with computer assistance**.

Math is now **data** that can be **processed**, **transformed**, and **inspected** in various ways.

Preamble: the Perfectoid Spaces Project (cont.)

Kevin Buzzard, Patrick Massot, Johan Commelin




mathoverflow

Home

What are "perfectoid spaces"?

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Here is a completely different kind of answer to this question.

A *perfectoid space* is a term of type `PerfectoidSpace` in the [Lean theorem prover](#).

Here's a quote from the source code:

```
structure perfectoid_ring (R : Type) [Huber_ring R] extends Tate_ring R : Prop :=
(complete  : is_complete_hausdorff R)
(uniform   : is_uniform R)
(ramified  : ∃ ϖ : pseudo_uniformizer R, ϖ^p ∣ p in R°)
(Frobenius : surjective (Frob R°/p))
```

Mathlib > RingTheory > Finiteness.lean

```lean
theorem FG.stabilizes_of_iSup_eq {M' : Submodule R M} (hM' : M'.FG) (N : ℕ →o Submodule R M)
    (H : iSup N = M') : ∃ n, M' = N n := by
  obtain ⟨S, hS⟩ := hM'
  have : ∀ s : S, ∃ n, (s : M) ∈ N n := fun s =>
    (Submodule.mem_iSup_of_chain N s).mp
      (by
        rw [H, ← hS]
        exact Submodule.subset_span s.2)
  choose f hf using this
  use S.attach.sup f
  apply le_antisymm
  · conv_lhs => rw [← hS]
    rw [Submodule.span_le]
    intro s hs
    exact N.2 (Finset.le_sup <| S.mem_attach ⟨s, hs⟩) (hf _)
  · rw [← H]
    exact le_iSup _ _
```

▼ Finiteness.lean:365:2

▼ Tactic state

1 goal

▼ **case** intro

```
R : Type u_1
M : Type u_2
inst✝² : Semiring R
inst✝¹ : AddCommMonoid M
inst✝ : Module R M
M' : Submodule R M
N : ℕ →o Submodule R M
H : iSup ⇑N = M'
S : Finset M
hS : span R ↑S = M'
f : { x // x ∈ S } → ℕ
hf : ∀ (s : { x // x ∈ S }), ↑s ∈ N (f s)
⊢ ∃ n, M' = N n
```

Mathlib > RingTheory > Finiteness.lean

```lean
theorem FG.stabilizes_of_iSup_eq {M' : Submodule R M} (hM' : M'.FG) (N : ℕ →o Submodule R M)
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      (by
        rw [H, ← hS]
        exact Submodule.subset_span s.2)
  choose f hf using this
  use S.attach.sup f
  apply le_antisymm
  · conv_lhs => rw [← hS]
    rw [Submodule.span_le]
    intro s hs
    exact N.2 (Finset.le_sup <| S.mem_attach ⟨s, hs⟩) (hf _)
  · rw [← H]
    exact le_iSup _ _
```

▼ Finiteness.lean:365:2

▼ Tactic state

1 goal

▼ **case** intro

```
R : Type u_1
M : Type u_2
inst✝² : Semiring R
inst✝¹ : AddCommMonoid M
inst✝ : Module R M
M' : Submodule R M
N : ℕ →o Submodule R M
H : iSup ⇑N = M'
S : Finset M
hS : span R ↑S = M'
f : { x // x ∈ S } → ℕ
hf : ∀ (s : { x // x ∈ S }), ↑s ∈ N (f s)
⊢ ∃ n, M'
```

M' : Submodule R M

Mathlib > RingTheory > Finiteness.lean

```
theorem FG.stabilizes_of_iSup_eq {M' : Submodule R M} (hM' : M'.FG) (N : ℕ →o Submodule R M)
```

Defs.lean ~/projects/mathlib4/Mathlib/Algebra/Module/Submodule - Definitions (1)

```
assert_not_exists DivisionRing

open Function

universe u'' u' u v w

variable {G : Type u''} {S : Type u'} {R : Type u} {M : Type v} {ι :

/-- A submodule of a module is one which is closed under vector oper
  This is a sufficient condition for the subset of vectors in the su
  to themselves form a module. -/
structure Submodule (R : Type u) (M : Type v) [Semiring R] [AddCommM
  AddSubmonoid M, SubMulAction R M : Type v
```

structure Submodule (R : Type u) (

▼Finiteness.lean:356:44

▼Expected type

```
R : Type u_1
M : Type u_2
inst✝⁴ : Semiring R
inst✝³ : AddCommMonoid M
inst✝² : Module R M
P : Type u_3
inst✝¹ : AddCommMonoid P
inst✝ : Module R P
f : M →ₗ[R] P
⊢ Type u_2
```

▶ All Messages (0)

Mathlib > Algebra > Module > Submodule > Defs.lean > Submodule

```
  This is a sufficient condition for the subset of vectors in the submodule
  to themselves form a module. -/
structure Submodule (R : Type u) (M : Type v) [Semiring R] [AddCommMonoid M] [Module R M] extends
  AddSubmonoid M, SubMulAction R M : Type v
```

Defs.lean ~/projects/mathlib4/Mathlib/Algebra/Group/Submonoid - Definitions (1)

```
add_decl_doc Submonoid.toSubsemigroup

/-- `SubmonoidClass S M` says `S` is a type of subsets `s ≤ M` that
and are closed under `(*)` -/
class SubmonoidClass (S : Type*) (M : outParam Type*) [MulOneClass M
  MulMemClass S M, OneMemClass S M : Prop

section

/-- An additive submonoid of an additive monoid `M` is a subset cont
  closed under addition. -/
structure AddSubmonoid (M : Type*) [AddZeroClass M] extends AddSubse
  /-- An additive submonoid contains `0`. -/
  zero_mem' : (0 : M) ∈ carrier
```

structure AddSubmonoid (M : Type

▼ Defs.lean:37:8

▼ Expected type

```
G : Type u''
S : Type u'
R✝ : Type u
M✝ : Type v
ι : Type w
R : Type u
M : Type v
inst✝² : Semiring R
inst✝¹ : AddCommMonoid M
inst✝ : Module R M
⊢ Type v
```

▶ All Messages (0)

The Challenge

In November of 2020, Peter Scholze posits the Liquid Tensor Experiment (LTE) challenge.

> *"I spent much of 2019* ***obsessed*** *with the proof of this theorem,* ***almost getting crazy over it****. In the end, we were able to get an argument pinned down on paper, but I think nobody else has dared to look at the details of this, and so I still have some small lingering doubts"*,
>
> Peter Scholze

The First Victory

Johan Commelin led a team with several members of the **Lean community and announced the formalization of the crucial intermediate lemma** that Scholze was unsure about, with only minor corrections, in **May 2021**.

> *“[T]his was precisely the kind of oversight I was worried about when I asked for the formal verification. […] The proof walks a fine line, so if some argument needs constants that are quite a bit different from what I claimed, it might have collapsed”*, Peter Scholze


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NEWS | 18 June 2021

Mathematicians welcome computer-assisted proof in ‘grand unification’ theory

Achieving the Unthinkable

The full challenge was completed in July 2022.

The team not only verified the proof but also simplified it.
Moreover, they did this without fully understanding the entire proof.

Johan, the project lead, reported that he could only see two steps ahead. **Lean was a guide**.

> *"The Lean Proof Assistant was really that: an assistant in navigating through the thick jungle that this proof is. Really, one key problem I had when I was trying to find this proof was that I was essentially unable to keep all the objects in my RAM, and I think the same problem occurs when trying to read the proof"*, Peter Scholze

Only the Beginning

Independence of the Continuum Hypothesis, Han and van Doorn, 2021

Sphere Eversion, Massot, Nash, and van Doorn, 2020-2022

Fermat's Last Theorem for regular primes, Brasca et al., 2021-2023

Unit Fractions, Bloom and Mehta, 2022

Consistency of Quine's New Foundations, Wilshaw and Dillies, 2022-2024

Polynomial Freiman-Ruzsa Conjecture (PFR), Tao and Dillies, 2023

Prime Number Theorem And Beyond, Kontorovich and Tao, 2024-ongoing

Carleson Project, van Doorn, 2024-ongoing

The Equational Theories Project, Tao, 2024

Fermat's Last Theorem (FLT), Buzzard, 2024-ongoing, community estimates it will take +1M LoC

Should we trust Lean?

Lean has a small trusted proof checker.

Do I need to trust the checker?

No, **you can export your proof**, and use external checkers. There are checkers implemented in C/C++, Rust, Lean, etc.

You can implement your own checker.

What did we learn?

Machine-checkable proofs enable a new level of **collaboration** in mathematics.

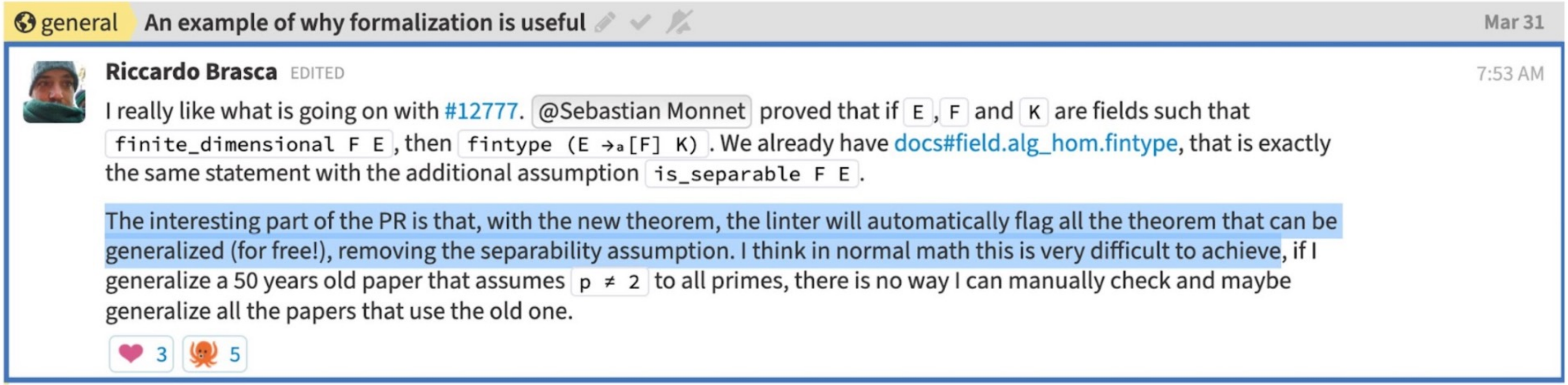
The power of the **community**.

We don’t need to trust our automation/moves.

It is not just about proving but also understanding complex objects and proofs, getting new insights, and navigating through the “thick jungles” that are **beyond our cognitive abilities**.

What did we learn?

Another unexpected benefit of formal mathematics: **auto refactoring** and **generalization**.


general | An example of why formalization is useful | Mar 31

Riccardo Brasca EDITED 7:53 AM

I really like what is going on with #12777. @Sebastian Monnet proved that if `E`, `F` and `K` are fields such that `finite_dimensional F E`, then `fintype (E →ₐ[F] K)`. We already have docs#field.alg_hom.fintype, that is exactly the same statement with the additional assumption `is_separable F E`.

The interesting part of the PR is that, with the new theorem, the linter will automatically flag all the theorem that can be generalized (for free!), removing the separability assumption. I think in normal math this is very difficult to achieve, if I generalize a 50 years old paper that assumes `p ≠ 2` to all primes, there is no way I can manually check and maybe generalize all the papers that use the old one.

❤ 3 🐙 5

Software

Lean in Software Verification

Lean is a programming language, and is used in **many software verification projects.**

You can write code and reason about it simultaneously.

You can prove that your code has the properties you expect.

> *“Testing can show the presence of bugs, but not their absence”*, E. Dijkstra

Differential Privacy

A mathematical framework that ensures the **privacy of individuals** in a dataset by adding controlled **random noise** to the data.

Discrete sampling algorithms, like the **Discrete Gaussian Sampler**, are used to add carefully calibrated noise to data.

What may go wrong if a buggy sampler is used?

Privacy Violations: leakage of sensitive information

Incorrect Results: distorted analysis results

SampCert

A project led by **Jean-Baptiste Tristan** at AWS.

An **open-source** Lean library of formally **verified differential privacy primitives**.

Tristan's implementation is not only verified, but it is also **twice as fast as the previous one**.

He managed to implement **aggressive optimizations** because Lean served as a guide, ensuring that **no bugs** were introduced.

SampCert would not exist without Mathlib

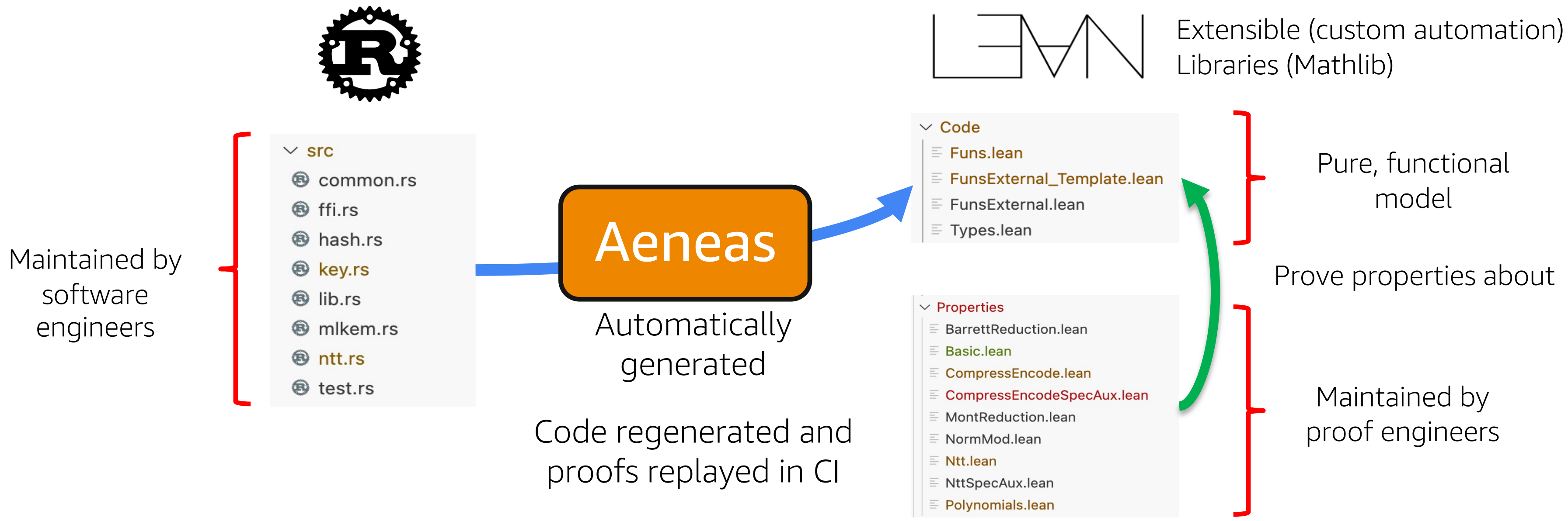
SampCert is software, but its verification relies heavily on Mathlib.

The verification of code addressing practical problems in data privacy depends on the formalization of mathematical concepts, from **Fourier analysis** to **number theory** and **topology**.

Verifying Cryptography with Aeneas at Microsoft

They verify (and fix/improve) the Rust code as written by software engineers

Code is evolving (new optimizations for specific hardware): They must adapt to rewrites



bv_decide: another powerful move

A verified bit-blaster by **Henrik Boving**, Josh Clune, Siddharth Bhat, and Alex Keizer

Uses LRAT proof producing SAT solvers: **Cadical**

```
/-
Close a goal by:
1. Turning it into a BitVec problem.
2. Using bitblasting to turn that into a SAT problem.
3. Running an external SAT solver on it and obtaining an LRAT proof from it.
4. Verifying the LRAT proof using proof by reflection.
-/
syntax (name := bvDecideSyntax) "bv_decide" : tactic
```

“Blasting” popcount with bv_decide

```
def popcount : Stmt := imp {
  x := x - ((x >>> 1) &&& 0x55555555);
  x := (x &&& 0x33333333) + ((x >>> 2) &&& 0x33333333);
  x := (x + (x >>> 4)) &&& 0x0F0F0F0F;
  x := x + (x >>> 8);
  x := x + (x >>> 16);
  x := x &&& 0x0000003F;
}
```

```
def pop_spec (x : BitVec 32) : BitVec 32 :=
  go x 0 32
where
  go (x : BitVec 32) (pop : BitVec 32) (i : Nat) : BitVec 32 :=
    match i with
    | 0 => pop
    | i + 1 =>
      let pop := pop + (x &&& 1#32)
      go (x >>> 1#32) pop i
```

```
theorem popcount_correct :
    ∃ ρ, (run (Env.init x) popcount 8) = some ρ ∧ ρ "x" = pop_spec x := by
  simp [run, popcount, Expr.eval, Expr.BinOp.apply, Env.set, Value, pop_spec, pop_spec.go]
  bv_decide
```

“Blasting” popcount with bv_decide

Imp.lean > { } Imp.Stmt > popcount_correct

```
theorem popcount_correct :
    ∃ ρ, (run (Env.init x) popcount 8) = some ρ
  simp [run, popcount, Expr.eval, Expr.BinOp.app
  bv_decide
```

▼Tactic state

1 goal

```
x : Value
⊢ ((x - (x >>> 1 &&& 1431655765#32) &&& 858993459#32) + ((x - (x >>> 1 &&&
1431655765#32)) >>> 2 &&& 858993459#32) +
            ((x - (x >>> 1 &&& 1431655765#32) &&& 858993459#32) +
                ((x - (x >>> 1 &&& 1431655765#32)) >>> 2 &&& 858993459#32)) >>>
              4 &&&
          252645135#32) +
        ((x - (x >>> 1 &&& 1431655765#32) &&& 858993459#32) +
                ((x - (x >>> 1 &&& 1431655765#32)) >>> 2 &&& 858993459#32) +
              ((x - (x >>> 1 &&& 1431655765#32) &&& 858993459#32) +
                  ((x - (x >>> 1 &&& 1431655765#32)) >>> 2 &&& 858993459#32)) >>>
                4 &&&
            252645135#32) >>>
          8 +
      (((x - (x >>> 1 &&& 1431655765#32) &&& 858993459#32) +
                ((x - (x >>> 1 &&& 1431655765#32)) >>> 2 &&& 858993459#32) +
              ((x - (x >>> 1 &&& 1431655765#32) &&& 858993459#32) +
                  ((x - (x >>> 1 &&& 1431655765#32)) >>> 2 &&& 858993459#32)) >>>
                4 &&&
            252645135#32) +
          ((x - (x >>> 1 &&& 1431655765#32) &&& 858993459#32) +
                  ((x - (x >>> 1 &&& 1431655765#32)) >>> 2 &&& 858993459#32) +
                ((x - (x >>> 1 &&& 1431655765#32) &&& 858993459#32) +
                    ((x - (x >>> 1 &&& 1431655765#32)) >>> 2 &&& 858993459#32)) >>>
                  4 &&&
              252645135#32) >>>
            8) >>>
        16 &&&
    63#32 =
  (x &&& 1#32) + (x >>> 1 &&& 1#32) + (x >>> 2 &&& 1#32) + (x >>> 3 &&& 1#32) + (x >>>
4 &&& 1#32) +
```

grind in Software Verification

```
example (x : BitVec 8) : (x + 16)*(x - 16) = x^2 := by
  grind +ring
```

```
def siftDown (a : Array Int) (root : Nat) (e : Nat) (h : e ≤ a.size := by grind) : Array Int :=
  if _ : leftChild root < e then
    let child := leftChild root
    let child := if _ : child+1 < e then
      if a[child] < a[child + 1] then child + 1 else child
    else child
    if a[root] < a[child] then
      let a := a.swap root child
      siftDown a child e
    else a
  else a
termination_by e - root

theorem siftDown_size {a root e h} : (siftDown a root e h).size = a.size := by
   fun_induction siftDown <;> grind [siftDown]
```

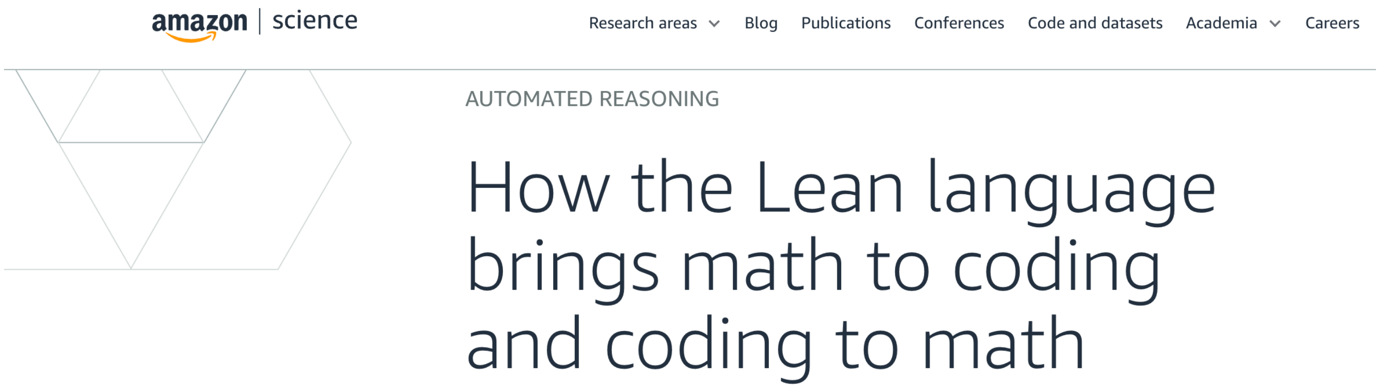
What did we learn?

Machine-checkable proofs enable you to **code without fear**.

Powerful proof automation.

Industrial projects: Verified compilers, policy languages, cryptographic libraries, etc.

Many more at the **Lean Project Registry**: https://reservoir.lean-lang.org/



AI

Lean Enables Verified AI for Mathematics and Code

LLMs are powerful tools, but they are prone to **hallucinations**.

In Math, a **small mistake can invalidate the whole proof**.

Imagine manually checking an AI-generated proof with the size and complexity of FLT.

The informal proof is **over 200 pages**.

Buzzard estimates a formal proof will require more than **1M LoC** on top of Mathlib.

Machine-checkable proofs are the antidote to hallucinations.

AI Proof Assistants

Several groups are developing AI that suggests the **next move**(s) in Lean’s interactive proof game.

LeanDojo is an open-source project from Caltech, and everything (model, datasets, code) is open.

OpenAI and Meta AI have also developed AI assistants for Lean.

The New York Times

Artificial Intelligence › | A.I.'s Math Problem | A.I. Training Data Disappears | Microsoft's Risk-Taker | Fine Print Changes | Quiz: Fake or Real Images?

Move Over, Mathematicians, Here Comes AlphaProof

A.I. is getting good at math — and might soon make a worthy collaborator for humans.

Share full article | 47

Ringing the gong at Google Deepmind's London headquarters, a ritual to celebrate each A.I. milestone, including its recent triumph of reasoning at the International Mathematical Olympiad. Google Deepmind

What did we learn?

Machine-checkable proofs enable **AI that does not hallucinate**.

LLMs are getting better and better at explaining Lean code.

In an era of big data and LLMs, machine-checkable proofs ensure trust in results.

AI systems that prove rather than guess.

Before we wrap up…

Lean Enables Decentralized Collaboration

Lean is Extensible

Users extend Lean using Lean itself.

Lean is implemented in Lean.

You can make it your own.

You can create your own moves.

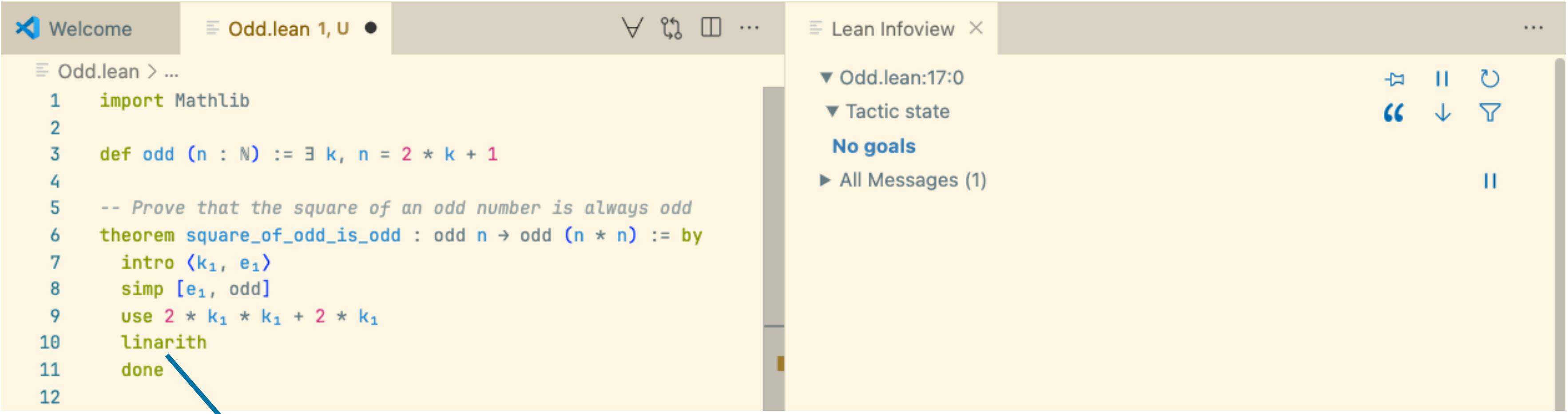
Machine-Checkable Proofs

You don’t need to trust me to use my proofs.

You don’t need to trust my automation to use it.

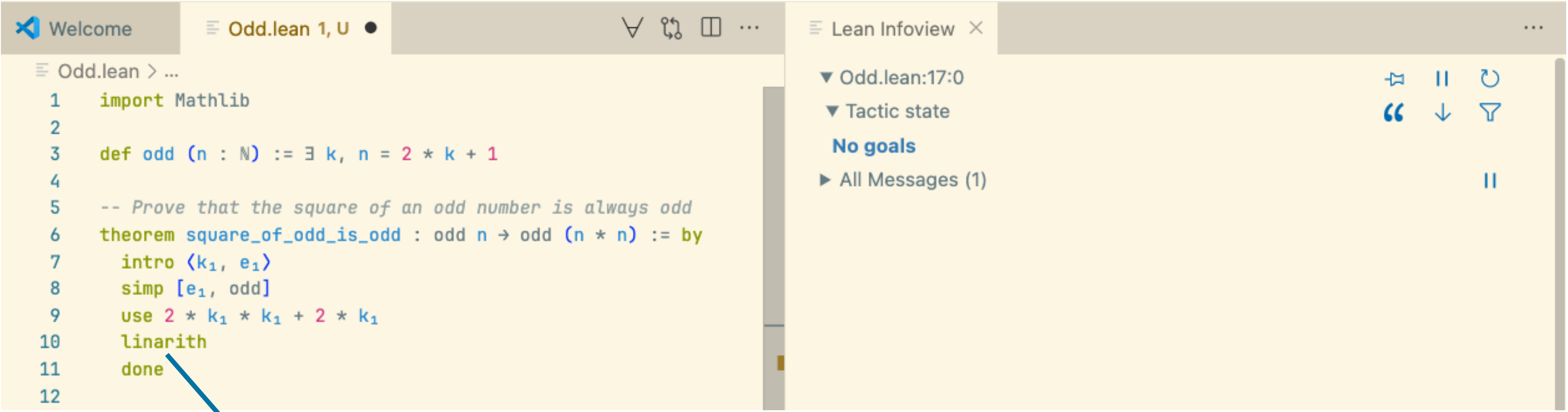
Code without fear.

Lean is a game where we can implement your own moves



The `linarith` “move” was implemented by the Mathlib community in Lean!

Lean is a game where we can implement your own moves



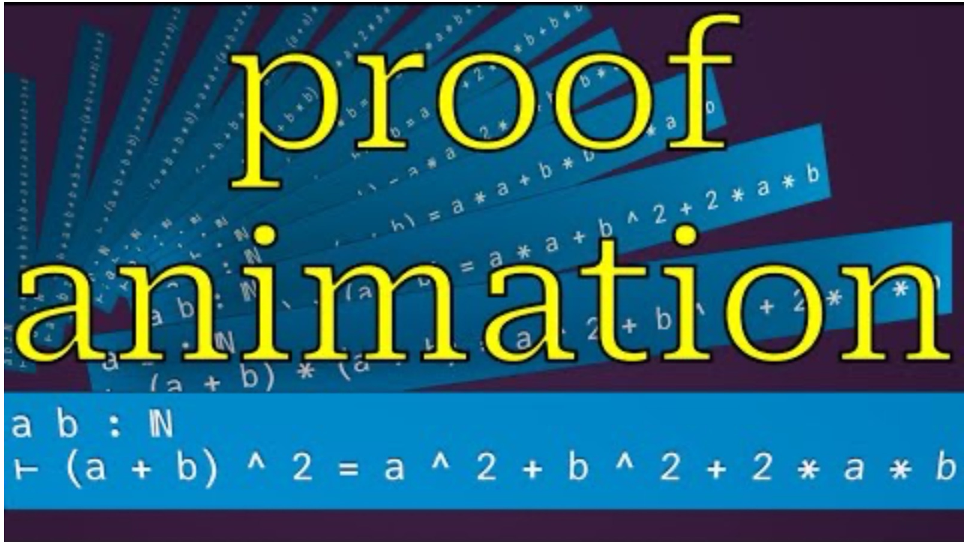
The `linarith` “move” was implemented by the Mathlib community in Lean!

The `bv_decide` and `grind` “moves” are also implemented in Lean!

You can use Lean to introspect its internal data

The tool lean-training-data is implemented in Lean itself. **It is a Lean package**.

A similar approach can be used to automatically generate proof animations.



Lean FRO: Shaping the Future of Lean Development

The Lean Focused Research Organization (FRO) is a non-profit dedicated to Lean's development.

Founded in **August 2023**, the organization has 19 members.

Its mission is to enhance critical areas: **scalability**, **usability**, **documentation**, and **proof automation**.

It must reach **self-sustainability in August 2028** and become the **Lean Foundation**.

Philanthropic support is gratefully acknowledged from the **Simons Foundation**, the **Alfred P. Sloan Foundation**, **Richard Merkin**, and **Alex Gerko**.

Lean FRO: by numbers

19 releases and **4,047 pull requests** merged in the main repository only since its launch in July 2023.

Public roadmaps: https://lean-fro.org/about/roadmap-y2/

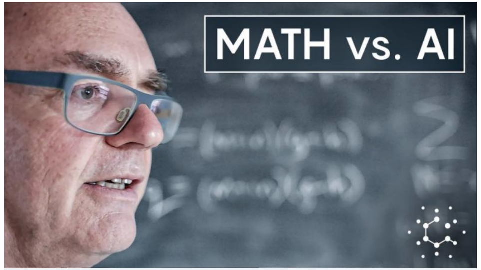
Lean project was featured in multiple venues NY Times, Quanta, Scientific American, etc.


The New York Times

A.I. and Chatbots › | Can A.I Be Fooled? | Testing a Tutorbot | Chatbot Prompts to Try | A.I.'s Literary Skills | What Are the Dangers of A.I.?

A.I. Is Coming for Mathematics, Too

For thousands of years, mathematicians have adapted to the latest advances in logic and reasoning. Are they ready for artificial intelligence?



When Computers Write Proofs, What's the Point of Mathematicians?

youtube.com

How can I contribute?

Help building Mathlib.

Want to engage with the vibrant Lean community? Join our Zulip channel.

Interested in ML kernels? Contribute to the KLR project.

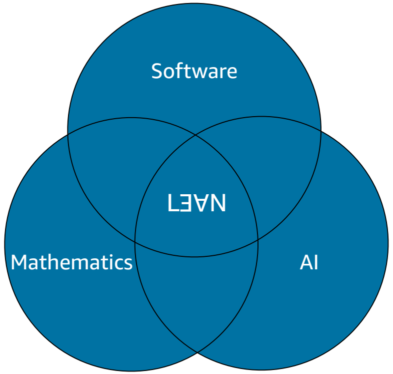
Want to contribute to a large formalization project? Join the FLT formalization project.

Start your own open-source Lean project! Your package will be available on our registry Reservoir.

Start using Lean online: live.lean-lang.org

Support the Lean FRO: Funding, partnerships, or simply advocating the project.

Conclusion



Lean is an **efficient programming language** and **proof assistant**.

The Mathlib community is changing how math is done.

It is not just about proving but also understanding complex objects and proofs, getting new insights, and navigating through the "thick jungles" that are **beyond our cognitive abilities**.

Lean tracks details, so humans focus on big ideas.

Decentralized collaboration with Lean: Large teams can collectively tackle huge proofs without losing track.

The entire discipline thrives when no one has to "take it on faith."

Thank You

https://leanprover.zulipchat.com/
x: @leanprover
LinkedIn: Lean FRO
Mastodon: @leanprover@functional.cafe
#leanlang, #leanprover

https://www.lean-lang.org/