On logics arising from the Čech-Stone compactification of ordinals.

Joel Lucero-Bryan, New Mexico State University

Joint work:
Guram Bezhanishvili, New Mexico State University
Nick Bezhanishvili, University of Amsterdam
Jan van Mill, University of Amsterdam

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Introduction

McKinsey-Tarski

Adopt an algebraic perspective of a topological space X:

$$(\wp(X), \mathbf{c}) \text{ where } \mathbf{c} : \wp(X) \to \wp(X) \text{ is closure in } X$$

Kuratowski closure axioms

$$\begin{aligned}
A \subseteq \mathbf{c}A \quad &\Leftrightarrow \quad p \to \Diamond p \\
\mathbf{cc}A \subseteq \mathbf{c}A \quad &\Leftrightarrow \quad \Diamond\Diamond p \to \Diamond p \\
\mathbf{c}(A \cup B) = \mathbf{c}(A) \cup \mathbf{c}B \quad &\Leftrightarrow \quad \Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q \\
\mathbf{c}\varnothing = \varnothing \quad &\Leftrightarrow \quad \text{(inference rule of necessitation)}
\end{aligned}$$

Interpreting the modal language: diamond is closure

A formula φ is valid in X if φ evaluates to X for each valuation assigning a subset to each proposition, written $X \models \varphi$.
The logic of X is $L(X) = \{\varphi \mid X \models \varphi\}$; notice $S4 \subseteq L(X)$.

Some classical results

- **[MT44]** S4 is the logic of any separable dense-in-itself metrizable space.
- **[RS63]** Under the axiom of choice, the M-T theorem above holds without the separability assumption.
- **[BGLB15]** The logics arising from metrizable spaces have been characterized, and include logics other than S4:

$$\mathsf{S4} \subseteq \mathsf{S4.1} \subseteq \mathsf{S4.Grz} \subseteq \cdots \subseteq \mathsf{S4.Grz}_2 \subseteq \mathsf{S4.Grz}_1$$

(these logics are defined subsequently)

How can we move beyond the realm of metrizable spaces?

A first answer... (via Kripke semantics)

- An S4-frame is an ordered pair $\mathfrak{F} = (W, R)$ where R is a quasi-ordering of W; that is, R is reflexive and transitive.
- $A \subseteq W$ is an R-upset of $\mathfrak{F}$ if $w \in A$ and wRv imply $v \in A$.
- The collection of R-upsets of $\mathfrak{F}$ form the Alexandroff topology τ_R on W; and closure is given by

$$\mathbf{c}A = R^{-1}(A) := \{w \mid (\exists v \in A)wRv\}.$$

- For each formula φ, we have that $\mathfrak{F} \models \varphi$ iff $(W, \tau_R) \models \varphi$.
- Each Kripke complete logic above S4 is topologically complete, but Alexandroff spaces are typically not metrizable (since they are rarely Hausdroff).

A motivating result for a second answer...

[BH09] The logic of the Čech-Stone compactification $\beta(\omega)$ of the ordinal space ω is S4.1.2, which obtained from S4 by postulating

$$\mathrm{ma} := \Box\Diamond p \to \Diamond\Box p \text{ and } \mathrm{ga} := \Diamond\Box p \to \Box\Diamond p.$$

Proof sketch...indicating some techniques

Sound:

- Call X densely discrete (DD) if its isolated points are dense. If X is DD, then $X \models \Box\Diamond p \to \Diamond\Box p$ (but not conversely).
- Call X extremally disconnected (ED) if the closure of each open set is open.
  X is ED iff $X \models \Diamond\Box p \to \Box\Diamond p$.
- $\beta(\omega)$ is both DD and ED, and thus $\mathrm{S4.1.2} \subseteq \mathrm{L}(\beta(\omega))$.

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Proof sketch...indicating some techniques

Complete:

- **Key tool:** A mapping $f : X \to Y$ between spaces is interior if f is continuous and open, and Y is an interior image of X if f is onto; we have $f^{-1}(\mathbf{c}B) = \mathbf{c}f^{-1}(B)$ for each $B \subseteq Y$.

  If Y is an interior image of X, then $\mathsf{L}(X) \subseteq \mathsf{L}(Y)$; thus, $Y \not\models \varphi$, then $X \not\models \varphi$ (including the case when Y is an Alexandroff space of an S4-frame; $f^{-1}(R^{-1}(B)) = \mathbf{c}f^{-1}(B)$).

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Proof sketch...indicating some techniques

Complete:

- Say (W, R) is rooted if there is a root $w \in W$ such that $R[w] := \{v \in W \mid wRv\}$.
  S4.1.2 is the logic of finite rooted S4-frames that have a unique maximal point.
- Construct an interior mapping $f : \beta(\omega) \to (W, \tau_R)$ for each such frame (W, R) above; then $\mathrm{L}(\beta(\omega)) \subseteq$ S4.1.2.
  The original proof uses set theoretic axioms beyond ZFC, but Dow has a proof in ZFC.

Shehtman's problems

Convention: View an ordinal as a topological space by equipping it with the interval topology induced by its order.

Generalizing the result that $L(\beta(\omega)) = S4.1.2$, Shehtman posed the following problems:

P1 For each $n \geq 1$, axiomatize the modal logic $L_n := L(\beta(\omega^n))$ of the Čech-Stone compactification $\beta(\omega^n)$ of the ordinal ω^n.

P2 Describe the logics that arise as the logic of $\beta(\gamma)$ for an arbitrary ordinal γ.

As we present a partial solution to **P2** we will see that the logics L_n appearing in **P1** play a key role in answering **P2**.

Preliminaries

Some known facts about ordinals

- Each ordinal γ is locally compact, normal, and scattered.
- Therefore, $\beta(\gamma)$ is a zero-dimensional DD space and γ is homeomorphic to an open subspace of $\beta(\gamma)$.

Definition

The logic S4.1 is obtained from S4 by postulating $\Box\Diamond p \to \Diamond\Box p$.

Lemma

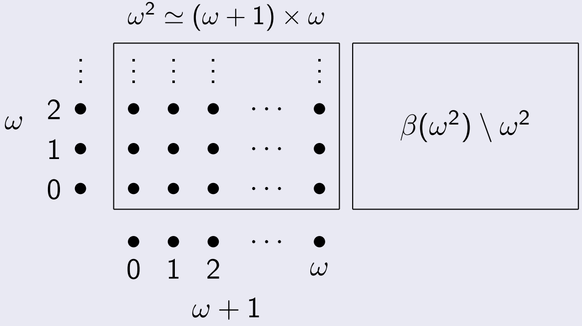
For any ordinal γ, we have that $\mathsf{S4.1} \subseteq \mathsf{L}(\beta(\gamma))$.

Proof

$\beta(\gamma) \models \Box\Diamond p \to \Diamond\Box p$ since $\beta(\gamma)$ is DD.

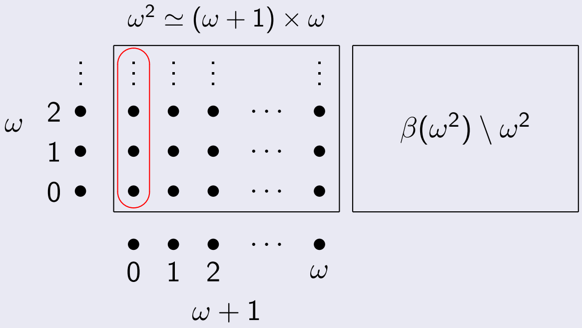
REALIZING ω^n

For $n \geq 1$, we work with the product space $(\omega^n + 1) \times \omega$ since it is homeomorphic to the ordinal ω^{n+1}, depicted below is $\beta(\omega^2)$:



Realizing ω^n

For $n \geq 1$, we work with the product space $(\omega^n + 1) \times \omega$ since it is homeomorphic to the ordinal ω^{n+1}, depicted below is $\beta(\omega^2)$:



In the red oval is a clopen copy of ω in ω^2.

Lemma

For each $n \geq 1$ we have that $L_{n+1} \subseteq L_n$, and hence

$$S4.1 \cdots \subseteq L_3 \subseteq L_2 \subseteq L_1 = S4.1.2$$

Pertinent known results

- If Y is an open subset of X, then $L(X) \subseteq L(Y)$.
- Let X be normal and $\mathbf{c}$ closure in $\beta(X)$.
  1. If Y is clopen in X, then $\mathbf{c}(Y)$ is clopen in $\beta(X)$.
  2. If Y is closed in X, then $\mathbf{c}(Y)$ is a compactification of Y equivalent to $\beta(Y)$.

Proof sketch

- ω^n is homeomorphic to a clopen subspace X of ω^{n+1}.
- $\beta(\omega^n)$ is equivalent to $\mathbf{c}X$, which is clopen in $\beta(\omega^{n+1})$.
- $L_{n+1} \subseteq L(\mathbf{c}X) = L_n$.

Lemma

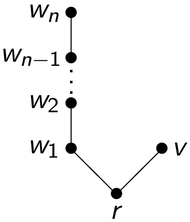
For each $n \geq 1$, we have $\mathsf{L}_{n+1} \subset \mathsf{L}_n$.

To prove this we require that each finite rooted S4-frame $\mathfrak{F} = (W, R)$ has its associated Fine-Jankov formula $\chi_{\mathfrak{F}}$ encoding $\mathfrak{F}$:

Topological version of Fine's result (known)

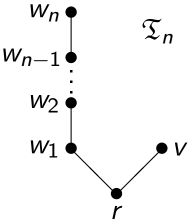
For each space X, we have that $X \models \chi_{\mathfrak{F}}$ iff $\mathfrak{F}$ is not an interior image of any open subspace of X.

Let χ_n be the Fine-Jankov formula of the partially ordered frame $\mathfrak{T}_n$ depicted below:



Lemma

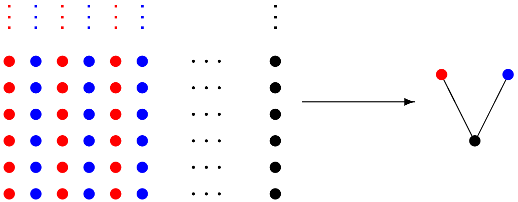
For each $n \geq 1$, we have $\mathsf{L}_{n+1} \subset \mathsf{L}_n$.



We have the following proof sketch that $\chi_n \notin \mathsf{L}(\beta(\omega^{n+1}))$:

- $\mathfrak{T}_n$ is an interior image of ω^{n+1} (picture proof for ω^2).
- $\beta(\omega^{n+1}) \not\models \chi_n$ since ω^{n+1} is homeomorphic to an open subset of $\beta(\omega^{n+1})$.
- $\chi_n \notin \mathsf{L}_{n+1}$.

Picture proof for ω^2 and $\mathfrak{T}_1$



Lemma

For each $n \geq 1$, we have $\mathsf{L}_{n+1} \subset \mathsf{L}_n$.

To prove this we additionally require the following:

Known

If Y and Z are disjoint closed subsets of a normal space X, then $\mathbf{c}(Y)$ and $\mathbf{c}(Z)$ are disjoint in $\beta(X)$.

Lemma

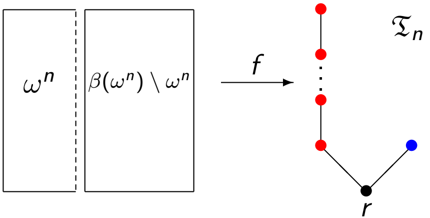
Let X be a zero-dimensional space and $\mathfrak{F} = (W, \leq)$ a rooted S4-frame containing a maximal point m. Then $\mathfrak{F}$ is an interior image of X iff $\mathfrak{F}$ is an interior image of some open subspace of X.

Corollary

Let $n \geq 1$ and $\mathfrak{F}$ a finite rooted S4-frame. Then $\beta(\omega^n) \models \chi_{\mathfrak{F}}$ iff $\mathfrak{F}$ is not an interior image of $\beta(\omega^n)$.

Lemma

For each $n \geq 1$, we have $\mathsf{L}_{n+1} \subset \mathsf{L}_n$.

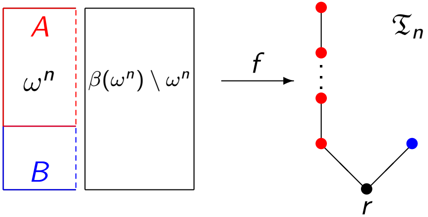


We have the following proof sketch that $\chi_n \in \mathsf{L}(\beta(\omega^n))$:

- Assume $f : \beta(\omega^n) \to \mathfrak{T}_n$ is an interior mapping. Let A be the preimage of the red subset in ω^n and B the preimage of the blue point in ω^n.

Lemma

For each $n \geq 1$, we have $\mathsf{L}_{n+1} \subset \mathsf{L}_n$.



We have the following proof sketch that $\chi_n \in \mathsf{L}(\beta(\omega^n))$:

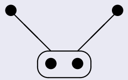
- Assume $\mathfrak{T}_n$ is an interior image of $\beta(\omega^n)$. Let A be the preimage of the red subset in ω^n and B the preimage of the blue point in ω^n.
- $\{A, B\}$ is a partition of ω^n consisting of open sets.
- Thus, $\varnothing \neq f^{-1}(r) \subseteq \mathbf{c}(A) \cap \mathbf{c}B = \varnothing$, which is a contradiction.
- Applying the corollary, we have $\beta(\omega^n) \models \chi_n$, or that $\chi_n \in \mathsf{L}_n$.

Thus we have established that

$$S4.1 \subseteq \cdots \subset L_3 \subset L_2 \subset L_1 = S4.1.2$$

Lemma

The S4.1-frame $\mathfrak{F}$ depicted below is not an interior image of $\beta(\omega^n)$ for any $n \geq 1$:



Thus, $\chi_{\mathfrak{F}} \in L(\beta(\omega^n))$ for each $n \geq 1$. (Proof is similar to previous)

$\chi_{\mathfrak{F}} \notin S4.1$ since $\mathfrak{F}$ is an S4.1-frame, which motivates defining

$$L_\infty = \bigcap_{n \geq 1} L_n, \text{ and hence}$$

$$S4.1 \subset L_\infty \subset \cdots \subset L_3 \subset L_2 \subset L_1 = S4.1.2$$

On to **P2** ...

We now recall a useful representation of a nonzero ordinal.

Definition

Each nonzero ordinal γ can be uniquely written in the Cantor normal form $\gamma = \omega^{\alpha_k} n_k + \cdots + \omega^{\alpha_1} n_1$, where $0 \neq k \in \omega$, each n_i is nonzero and finite, and $0 \leq \alpha_1 < \cdots < \alpha_k$ are ordinals.

Theorem (Structural Theorem)

For an infinite ordinal γ, we have that $\beta(\gamma)$ is homeomorphic to the disjoint union of a compact ordinal and the Čech-Stone compactification of a power of ω.

Proof sketch of structural theorem

We only consider the case when γ is not compact:

- Write $\gamma = \omega^{\alpha_k} n_k + \cdots + \omega^{\alpha_1} n_1$ in Cantor normal form ($\alpha_1 \neq 0$ as γ is not compact).
- "Tear off" one copy of ω^{α_1} from "the top" of γ by writing $\gamma = \gamma' + \omega^{\alpha_1}$
- Basic ordinal arithmetic yields that $\gamma = \gamma' + \omega^{\alpha_1} = \gamma' + (1 + \omega^{\alpha_1}) = (\gamma' + 1) + \omega^{\alpha_1}$
- $\{\gamma' + 1, \omega^{\alpha_1}\}$ is a clopen partition of γ, implying $\{\gamma' + 1, \mathbf{c}(\omega^{\alpha_1})\}$ is a clopen partition of $\beta(\gamma)$ (as $\gamma' + 1$ is compact).
- $\mathbf{c}(\omega^{\alpha_1})$ is homeomorphic to $\beta(\omega^{\alpha_1})$.
- $\beta(\gamma)$ is homeomorphic to the disjoint union $(\gamma' + 1) \oplus \beta(\omega^{\alpha_1})$.

Preparing to use the structural theorem

Definition

S4.Grz is obtained from S4 by postulating the Grzegorczyk axiom

$$\Box(\Box(p \to \Box p) \to p) \to p$$

$\mathrm{S4.Grz}_n$ is obtained from S4.Grz by postulating $\mathrm{bd_n}$ where

$$\begin{aligned} \mathrm{bd}_1 &= \Diamond\Box p_1 \to p_1 \\ \mathrm{bd}_{n+1} &= \Diamond(\Box p_{n+1} \wedge \neg \mathrm{bd}_n) \to p_{n+1} \end{aligned}$$

Theorem (Abashidze-Blass)

Let $\gamma \neq 0$ be an ordinal and $n \geq 1$.

1. *If $\omega^{n-1} < \gamma \leq \omega^n$, then $\mathrm{L}(\gamma) = \mathrm{S4.Grz}_n$.*
2. *If $\omega^\omega \leq \gamma$, then $\mathrm{L}(\gamma) = \mathrm{S4.Grz}$.*

Employing the structural theorem

Let γ be a noncompact infinite ordinal.

Useful fact

If X is the disjoint union $Y \oplus Z$, then $\mathsf{L}(X) = \mathsf{L}(Y) \cap \mathsf{L}(Z)$.

Combining this with the structural theorem yields:

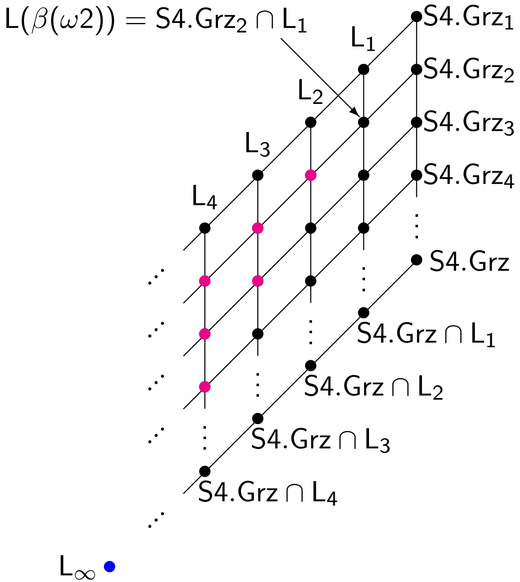
$$\mathsf{L}(\beta(\gamma)) = \mathsf{L}\left((\gamma' + 1) \oplus \beta(\omega^{\alpha_1})\right) = \mathsf{L}(\gamma' + 1) \cap \mathsf{L}(\beta(\omega^{\alpha_1}))$$

Thus, when $\alpha_1 = n$ is finite we have $\mathsf{L}(\beta(\gamma)) = \mathsf{L}(\gamma' + 1) \cap \mathsf{L}_n$, and using the Abashidze-Blass theorem gives the result.

Example

$\mathsf{L}(\beta(\omega 2)) = \mathsf{L}\left((\omega + 1) \oplus \beta(\omega)\right) = \mathsf{L}(\omega + 1) \cap \mathsf{L}_1 = \mathsf{S4.Grz}_2 \cap \mathsf{S4.1.2}$

Logics of the form $L(\beta(\gamma))$–a picture



A partial solution of **P2**–explicit description

Theorem

Each of the following logics arises as $L(\beta(\gamma))$ *for some ordinal* γ.

1. L_m *for each* $0 < m < \omega$.
2. S4.Grz *and* $S4.Grz_n$ *for each* $0 < n < \omega$.
3. $S4.Grz \cap L_m$ *and* $S4.Grz_n \cap L_m$ *for* $0 < m < n < \omega$.

All these logics are depicted by black bullets in the previous figure.

Theorem

Let γ *be a nonzero ordinal. If* $\beta(\gamma)$ *is an ordinal or* α_1 *in the Cantor normal form of* γ *is finite, then* $L(\beta(\gamma))$ *is a logic appearing as a black bullet in the previous figure.*

Conjecture

If γ is not compact and $\alpha_1 \geq \omega$ in its CNF, then $L(\beta(\gamma)) = L_\infty$.

Thank you for your attention ... any questions?

This work is available at: https://doi.org/10.1112/jlms.70090

Upcoming attractions: the axiomatization of L_2.