Relative Δ_3 Categoricity for Linear Orderings and Enumerative Combinatorics

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Relative Categoricity Level

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The categoricity level is closely tied to the **Scott rank** of the structure.

The simplest linear orderings

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Theorem: [Remmel] The relatively computably categorical (i.e., categoricity rank 1) linear orderings are exactly those with finitely many successivities. E.g. η, $\eta + 3 + \eta + 2$ etc.

Theorem: [McCoy] The relatively Δ_2 categorical (i.e., categoricity rank 2) linear orderings are those that are separated, finite sums of

$$\{\omega, \omega^*, \omega + \omega^*, \{k\}_{k\in\omega}, \{k \cdot \eta\}_{k\in\omega}\}.$$

Categoricity rank 3?

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Theorem: [G., Rossegger] There is a computable functional that transforms any computably categorical structure into a linear ordering with categoricity rank 3.

A solution: homogeneity

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To get further along the hierarchy: Enrich your structure with additional definable relations or functions

sp-homogeneity

Definition: In a linear ordering $(L, <)$ let $s : L \to L$ be defined by $s(x)$ is the successor of x if it exists and x otherwise. $p : L \to L$ is the analagously defined predecessor function.

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Examples: ω , $Sh(A)$ for $A \subseteq \omega \cup \{\omega, \omega^*, \zeta\}$, $\eta + 2 + 3 \cdot \eta + 4 + 5 \cdot \eta + 6 + 7 \cdot \eta + \cdots$.

sp-homogeneity fundamental facts

Proposition: [CCGHN] All weakly sp-homogeneous linear orderings are relatively Δ_4 categorical.

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Questions: Which linear orderings are sp-homogeneous? Which sp-homogeneous orderings have categoricity rank 3?

Main Computability results

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Theorem:[CCGHN] An sp-homogeneous linear ordering L is relatively Δ_3 categorical if and only if

1. It has no intervals of the form $Sh(S)$ where S includes an infinite block and finite blocks of arbitrary size or ζ along with another infinite block.
2. If I is $\omega \cdot \eta$, $Sh(\omega, \omega^*)$, $\omega^* \cdot \eta$ or $\zeta \cdot \eta$ or a sum of at least two of those orderings, it has an interval to its left and right in L that only has finite blocks of a bounded size.
3. Any $\zeta \cdot \eta$ does not have an interval to its right or left isomorphic to a shuffle sum including an infinite block.

How does this at all relate to enumerative combinatorics?

The Structure of sp-homogeneity

Theorem:[CCGHN] All elements in an sp-homogeneous linear ordering either lie in a unique block type or an interval isomorphic to a shuffle sum - block types are not re-used.

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Theorem:[CCGHN] weakly sp-homogeneous linear orderings are finite alternating sums of individual blocks and sp-homogeneous linear orderings.

Transforming sp-homogeneous Orderings

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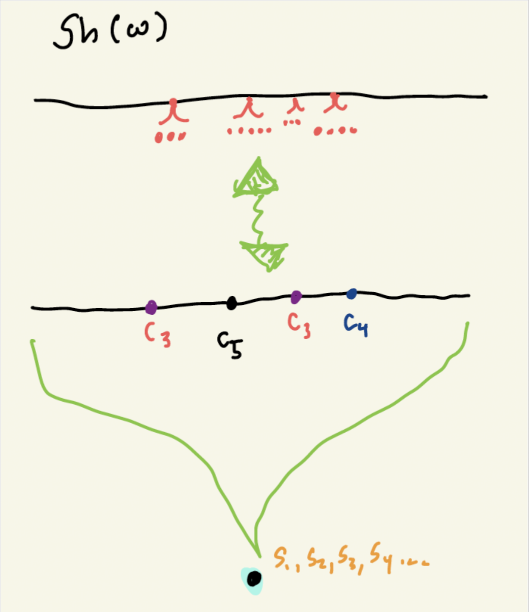
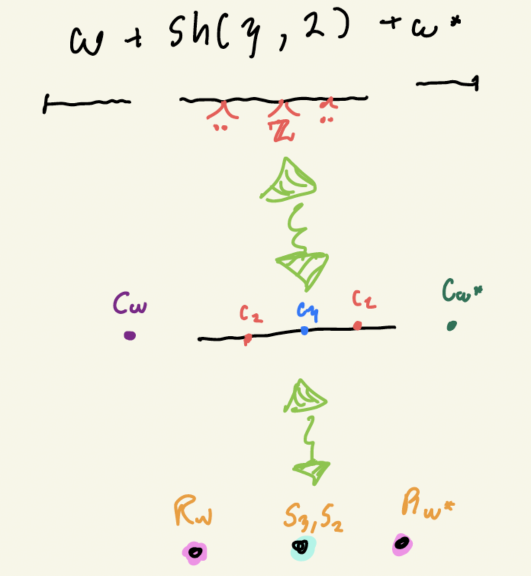
Homogeneous colored linear orderings can be turned into linear orderings with predicates by collapsing shuffle sums to one point and using predicates to remember which colors were shuffled in

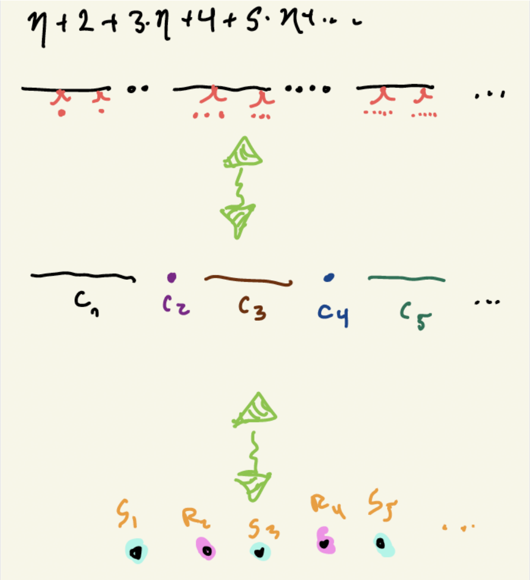
Example 1



Example 2



Example 3



Stronger Homogeneity Notions

Definition: A linear ordering L is $C_{n,m}$ homogeneous if when L is expanded to included definitions for, $\{S_i\}_{i<n}$, $\{P_j\}_{j<m}$, and $\{Adj_k\}_{k<n+m}$ it becomes homogeneous.

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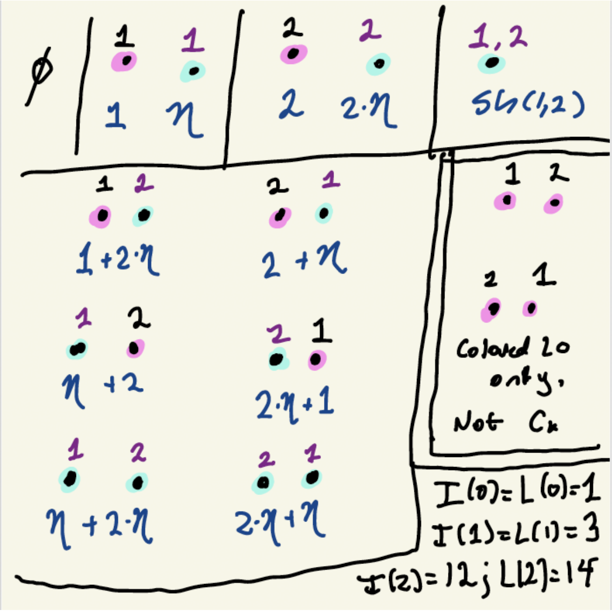
Theorem:[CCGHN] $C_{\infty,\infty}$ homogeneous coincides sp-homogeneous.

Theorem:[CCGHN] $C_{n,m}$ homogeneous depends only on $k = m + n + 1$ and coincides with homogeneous linear orderings with k colors with no adjacent singletons

$k = 2$



ϕ
1 η
2 2·η
1,2
Sh(1,2)
1+2·η
2+η
η+2
2·η+1
η+2·η
2·η+η
Colored 2o only.
Not Ck
I(0)=L(0)=1
I(1)=L(1)=3
I(2)=12; L(2)=14

General Combinatorial Questions

Question: What is the value of $I(k)$, the number of $C_{n,m}$ homogeneous linear orderings with $k = n + m + 1$?

Question: What is the value of $L(k)$, the number of homogeneous linear orderings with k colors?

Question: How do the growth rates compare?

$L(k)$

Theorem: $L(k)$ are the coefficients in the exponential generating function

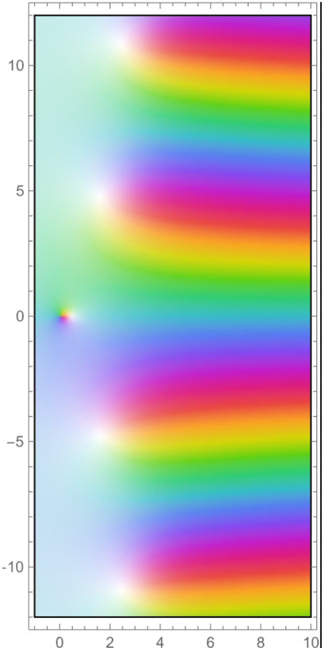
$$H(x) = \frac{e^x}{2 - x - e^x}.$$

Furthermore,

$$L(k) \sim -k! R \left(\frac{1}{Z}\right)^{k+1},$$

where $Z = 2 - W(e^2) \approx 0.442854$ and
$R = \frac{-e^2}{e^{W(e^2)} + e^2} \approx -0.6089389$.

$$\frac{e^x}{2-x-e^x}$$



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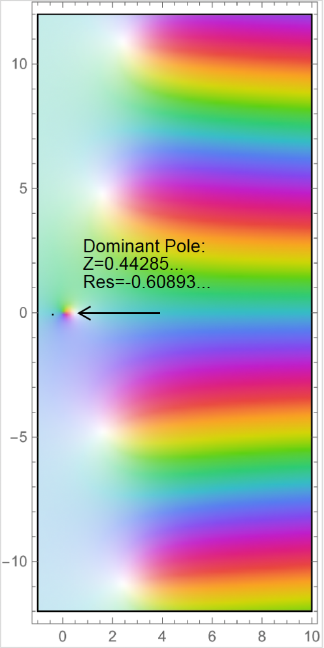

I(k)

Theorem: Let $I(k)$ be the number of $C_{n,m}$-homogeneous linear orderings with $k = n + m + 1$.

$$I(k) = \sum_{m=1}^{k} \binom{k}{m} \sum_{n=1}^{m} \sum_{r=0}^{\lceil \frac{n}{2} \rceil} \binom{n-r+1}{r} \binom{m}{r} r!(n-r)!S(m-r, n-r),$$

Where $S(n, m)$ are the Stirling numbers of the second kind. Furthermore, $I(k) = O(k!2.123^k)$.

$$\left(\frac{1}{ln(2)}\right)^{(1-p)} 2^{(1-p)H(\frac{p}{1-p})}$$

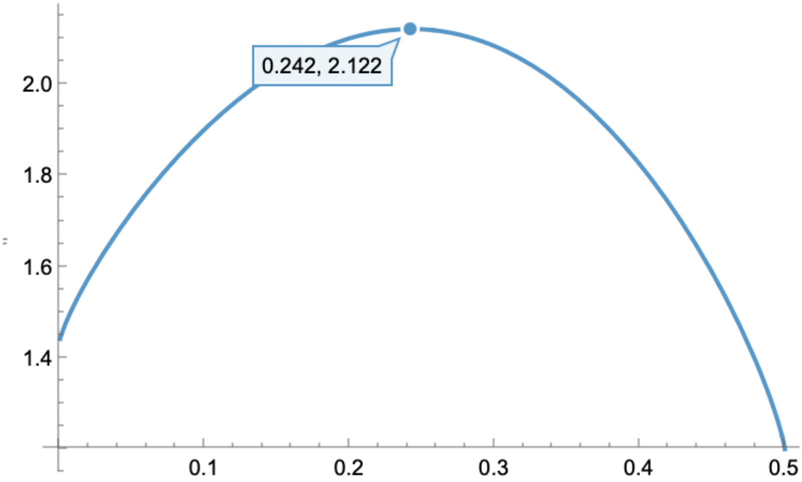

Figure: $H(x) = -x\log_2(x) - (1-x)\log_2(1-x)$

Thank you!