Differential Galois theory allowing additional algebraic constants

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Differential Fields

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We focus on the ordinary (one derivation) case and characteristic 0.

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From atomicity of the prime model: $C_{K^{\mathrm{diff}}} = (C_K)^{\mathrm{alg}}$.

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Ex. Let $K = (\mathbb{Q}, \frac{d}{dt}) \subset (\mathbb{Q}(e^t), \frac{d}{dt}) = L$. Then e^t is a $\mathbb{Q}$-basis of $V(L) = \{ce^t : c \in C_L = C_K = \mathbb{Q}\}$ for $\delta(y) - y = 0$.

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Since $C_{K^{\text{diff}}} = (C_K)^{\text{alg}}$, then $C_L = C_K$ for any intermediate L:
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Existence and uniqueness/multiplicity of Picard-Vessiot extensions is sensitive to field-arithmetic, model-theoretic, and Galois-cohomological properties of C_K.

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3. The torsor theorem: The realizations of $\text{tp}(\bar{b}/K(C_{K^{\text{diff}}})) = \text{tp}(\bar{b}/K)$ in K^{diff}, $Q_{\bar{b}}(K^{\text{diff}})$, is a right K-definable torsor for $G(C_{K^{\text{diff}}})$.

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(Epstein 1955a): One can always find a fundamental system $\bar{b}$ for Y, an OHLDE, such that C_L/C_K is a Galois extension. (Epstein 1955b): A partial Galois correspondence is shown for these extensions $L = K\langle \bar{b} \rangle$ with C_L/C_K Galois.

PV extensions with new algebraic constants

Theorem (M. 2025), Generalizes (Epstein 1955b): Theorem 1.

Let L/K be generated by a fundamental system with C_L/C_K a finite algebraic extension. Then $\mathrm{Aut}(L/K) \cong G(C_K)$ for G a linear algebraic group defined over C_K.

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Proposition (M. 2025).

Moreover F' is fixed under the direct correspondence if and only if $H_{F'}(C_L)$ is Zariski dense in $H_{F'}$.

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We can define a definable groupoid action living on Z and X_1 as follows.

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Elements $b \in Q_{\bar{d}}(K^{\mathrm{diff}})$, by the definitions, generate generalized strongly normal extensions exactly when $\bar{d} \in X(K)$. Then both H^+ and $H_{\bar{d}}$ are K-definable and are the intrinsic and extrinsic differential Galois groups of the extension L/K where $L = \mathrm{dcl}(K, b) = K\langle b \rangle$.

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Theorem (M. 2025).

The group $G(C_K)$ from Ep55b can be recovered from the Galois groupoid:

$$G(C_K) \cong (\sqcup_{d_1, d_2 \in \Lambda} H_{d_1,d_2}(C_L) \times H_0(C_K), *_{\bar{d}})$$

defining $(\alpha_1, \beta_1) *_{\bar{d}} (\alpha_2, \beta_2) := (\alpha_1 \beta_1(\alpha_2), \beta_1 \beta_2)$.

Relationship to Umemura's quasi-automorphic extensions

Defn (Umemura 1996):

A finitely generated differential extension L/K in K^{diff} is called quasi-automorphic if $L = K\langle \bar{b} \rangle$ for some $\bar{b} \in X(K^{\text{diff}})$ for X a K-definable right PHS for $G(C_K^{\text{diff}})$ an algebraic group defined over C_K.

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L/K is quasi-automorphic and $C_L = C_K$ iff K/L is strongly normal. [Easy to observe although not stated L/K is linear quasi-automorphic with $C_L = C_K$ iff L/K is PV.]

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Final Remarks. 1. The model (X,G) of a quasi-automorphic extension is not uniquely determined. 2. Umemura's automorphic extensions do not yet have a satisfactory model-theoretic definition, they generalize Hopf Galois extensions. 3. Umemura has a parameterized version.

Thank you!

Selected references

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