Countable Ordered Groups and Weihrauch Reducibility

Ang Li

University of Wisconsin–Madison

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Reverse mathematics and the big five

Reverse mathematics studies the axiomatic strength needed to prove theorems of ordinary mathematics over a weak base theory. It is usually studied using subsystems of second order arithmetic.

1. RCA_0: $\mathsf{PA}^- + \mathsf{I}\Sigma^0_1 + \Delta^0_1$-$\mathsf{CA}$
2. WKL_0: RCA_0 + some form of weak könig lemma
3. ACA_0: RCA_0 + arithmetical comprehension axiom
4. ATR_0: ACA_0 + arithmetical transfinite recursion scheme
5. Π^1_1-CA_0: RCA_0 + Π^1_1-comprehension axiom

Theorem

The following are equivalent over RCA_0*:*

1. Π^1_1-CA_0
2. *For any sequence of trees* $\langle T_k : k \in \mathbb{N}\rangle$, $T_k \subseteq \mathbb{N}^{<\mathbb{N}}$, *there exists a set* X *such that* $\forall k(k \in X \leftrightarrow T_k$ *has a path*$)$.

Order Type of Countable Ordered Groups

Theorem (Maltsev, 1949)

The order type of a countable ordered group is $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$, where α is an ordinal and $\varepsilon = 0$ or 1.

Definition

An ordered group is a pair $(G, \leq_G)$, where G is a group, $\leq_G$ is a linear order on G, and for all $a, b, g \in G$, if $a \leq b$ then $ag \leq bg$ and $ga \leq gb$.

For $\mathbb{Z}^\alpha$ and products of linear orders, we order by the rightmost coordinate on which two elements differ.

Theorem (Solomon, 2001)

The following are equivalent under RCA_0*:*

1. Π^1_1-CA_0
2. *If G is a countable ordered group, there is a well-order α and $\varepsilon \in \{0, 1\}$ such that $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$ is the order type of G.*

Theorems as problems

Statements like the ones in the previous theorem can be written as follows:

$$(\forall x \in X)(\exists y \in Y)[\varphi(x) \to \psi(x, y)].$$

We can naturally translate it to a computational problem, i.e., given an input x such that $\varphi(x)$, the output is y such that $\psi(x, y)$.

For our purposes, we consider problems on Baire space $\mathbb{N}^{\mathbb{N}}$, i.e., relations $f \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, or equivalently partial multi-valued functions $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$.

Remark: for many statements, there could be multiple natural ways to phrase them as a computational problem.

Weihrauch reducibility

Definition

Let f, g be partial multi-valued functions on Baire space. f is Weihrauch reducible to g, denoted $f \leq_{\mathrm{W}} g$ if there are computable Φ, Ψ on Baire space such that:

- given $p \in \mathrm{dom}(f)$, $\Phi(p) \in \mathrm{dom}(g)$, and
- given $q \in g(\Phi(p))$, $\Psi(p,q) \in f(p)$.

Φ, Ψ are called forward functional and backward functional respectively.

$$
\begin{array}{ccc}
p & \xrightarrow{\Phi} & \Phi(p) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
f(p) & \xleftarrow{\Psi(p,\cdot)} & q
\end{array}
$$

Algebraic operations

Definition

Among the many operations in Weihrauch degrees, we will use the following:

1. compositional product $f * g$: allows us to "use" g first, then f
2. product $f \times g$: allows us to use both f and g in parallel
3. finite parallelization f^*: allows us to use f finitely many times in parallel
4. parallelization $\widehat{f}$: allows us to use f countably many times in parallel

Big five and Weihrauch reducibility

There are imperfect analogies between the big five and Weihrauch degrees:

1. RCA_0: $\mathsf{Id}_{\mathbb{N}^\mathbb{N}}$.
2. WKL_0: $\mathsf{C}_{2^\mathbb{N}}$.
3. ACA_0: iterations of lim.
4. ATR_0: many candidates, $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$, $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$, etc.
5. Π^1_1-CA_0: $\widehat{\mathsf{WF}}$.

Definition

$\mathsf{C}_{\mathbb{N}^\mathbb{N}}$: Given an ill-founded tree in Baire space, find a path through it.

WF: Given a tree in Baire space, tell whether it is well-founded.

A small part of the zoo

$$
\begin{tikzcd}
\widehat{\mathsf{WF}} \arrow[d] \arrow[ddr] & \\
\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \arrow[d] & \\
\lim \equiv_{\mathrm{W}} \widehat{\mathsf{LPO}} \arrow[d] & \mathsf{WF} \arrow[dddl] \\
\mathsf{C}_{\mathbb{N}} \arrow[d] \arrow[dr] & \\
\lim_2 \arrow[d] & \mathsf{Min} \equiv_{\mathrm{W}} \mathsf{LPO}^* \arrow[dl] \\
\mathsf{Id}_{\mathbb{N}^{\mathbb{N}}} &
\end{tikzcd}
$$

Weihrauch problems

We have choices to make when formulating Malstev Theorem into Weihrauch problems.

1. $\mathsf{OG} \mapsto \alpha\varepsilon\mathsf{f}$: given a countable ordered group G, output the ordinal α and $\varepsilon \in \{0, 1\}$ in its order type $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$ with an order-preserving function from G to $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$.
2. $\mathsf{OG} \mapsto \alpha\varepsilon$

As part of our analysis, we also consider the following problems:

3. $\mathsf{OG} \mapsto \varepsilon$
4. $\mathsf{OG} \mapsto \alpha$
5. $\mathsf{OG} \mapsto \alpha\mathsf{0}$
6. $\mathsf{OG}\alpha\varepsilon \mapsto \mathsf{f}$: given a countable ordered group G with the ordinal α and $\varepsilon \in \{0, 1\}$ in its order type $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$, output an order-preserving function from G to $\mathbb{Z}^\alpha\mathbb{Q}^\varepsilon$.

Output everthing

Proposition

$\mathsf{OG} \mapsto \alpha\varepsilon\mathsf{f} \geq_{\mathrm{W}} \widehat{\mathsf{WF}}$

What about the other direction? Suppose we are given a computable ω-copy of the group, the output we need includes an ω-copy of the ordinal α. It is natural to ask: what can α be?

Definition

For any a, b in an ordered group G, they are Achimedean equivalent if there exists $m, n \in \mathbb{N}$ such that $|a^n| \geq |b|$ and $|b^m| \geq |a|$. Denoted, $a \approx b$. Also, a is Archimedean less than b if no such n exists. Denoted $a \ll b$.

Given an ω-copy of G,
$\mathrm{Arch}(G) := \{g \in G : (\forall h \in G)[h <_{\mathbb{N}} g \to \neg(h \approx g)]\}$, and
$W(\mathrm{Arch}(G))$ is the well-ordered initial segment of $\mathrm{Arch}(G)$.

What can α be?

Lemma

- *If the order type of G is* $\mathbb{Z}^\alpha$, $\mathrm{Arch}(G)$ *is a copy of* α. *If the order type of G is* $\mathbb{Z}^\alpha\mathbb{Q}$, $W(\mathrm{Arch}(G))$ *is a copy of an ordinal* $\beta \geq \alpha$.
- *If G is an ordered group, then we can uniformly compute* $\mathrm{Arch}(G)$ *from* G'.
- *There is a group with order type* $\mathbb{Z}^{\mathrm{H}}$ *where* $\mathrm{H} \cong \omega_1^{\mathrm{CK}}(1+\mathbb{Q})$ *is the Harrison linear order.*

Theorem

- *If a computable ordered group has order type* $\mathbb{Z}^\alpha$, *then* α *is computable.*
- *If a computable ordered group has order type* $\mathbb{Z}^\alpha\mathbb{Q}$, *then* $\alpha \leq \omega_1^{\mathrm{CK}}$.
- *There exists a computable countable ordered group with order type* $\mathbb{Z}^{\omega_1^{\mathrm{CK}}}\mathbb{Q}$.

$\mathsf{OG} \mapsto \alpha\varepsilon\mathsf{f} \leq_{\mathsf{W}} \widehat{\mathsf{WF}}$

About this proof:

- $\mathsf{OG} \mapsto \varepsilon \equiv_{\mathsf{W}} \mathsf{WF}$.
- In the reverse math proof, Π^1_1-CA_0 was used twice. The second application is relative to the first.
- $\widehat{\mathsf{WF}}$ is not closed under composition.
- To apply $\widehat{\mathsf{WF}}$ only once, the backward functional needs to approximate α and build the partial order-preserving map according to the current guess of α at the same time.

On the side

If we do not output the isomorphism f, then our Weihrauch problems turn out to be incomparable with fairly weak problems.

Proposition

- $\mathsf{OG} \mapsto \alpha \not\geq_{\mathsf{W}} \mathsf{lim}_2$.
- $\mathsf{OG} \mapsto \alpha\varepsilon \not\geq_{\mathsf{W}} \mathsf{lim}_2 \times \mathsf{lim}_2$.

Definition

$\mathsf{lim}_2 :\subseteq 2^{\mathbb{N}} \to 2$ is the limit operation on $\{0, 1\}$.

We will show this by looking at the first-order parts of these problems.

First-order part

Definition (Dzhfarov, Solomon, & Yokoyama)

A problem P is first-order if the codomain of P is $\mathbb{N}$.
We let $\mathcal{F}$ denote the class of all first-order problems.

For every Weihrauch problem P, there is a first order problem ${}^1\mathsf{P}$ witnessing the existence of $\max_{\leq_W}\{\mathsf{Q} \in \mathcal{F} : \mathsf{Q} \leq_W \mathsf{P}\}$. It is called the first-order part of P.

First-order part

Definition

$$\mathsf{LPO} : \mathbb{N}^{\mathbb{N}} \to \{0,1\},\ \mathsf{LPO}(p) = \begin{cases} 0 \text{ if } (\exists k)[p(k) = 0], \\ 1 \text{ otherwise.} \end{cases}$$

Theorem (Brattka, Gherardi, Marcone, & Pauly)

- $\mathsf{LPO}^* \equiv_{\mathrm{W}} \mathsf{Min}$.
- $\mathsf{lim}_2 |_{\mathrm{W}} \mathsf{LPO}^*$.
- $\mathsf{lim}_2, \mathsf{LPO}^* <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$.

Proposition

- ${}^1\mathsf{OG} \mapsto \alpha \equiv_{\mathrm{W}} \mathsf{LPO}^*$.
- ${}^1\mathsf{OG} \mapsto \alpha\varepsilon \equiv_{\mathrm{W}} \mathsf{LPO}^* \times \mathsf{WF}$.

Corollary

$\mathsf{OG} \mapsto \alpha \not\geq_{\mathrm{W}} \mathsf{lim}_2$. $\mathsf{OG} \mapsto \alpha\varepsilon \not\geq_{\mathrm{W}} \mathsf{lim}_2 \times \mathsf{lim}_2$.

How much is needed to output α?

Proposition

$\mathsf{OG}\alpha\varepsilon \mapsto \mathsf{f} \leq_{\mathsf{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$

Idea: build a back-and-forth tree whose paths represent isomorphism between $\mathbb{Z}^{\alpha}\mathbb{Q}^{\varepsilon}$ and the group.

Corollary

$\mathsf{OG} \mapsto \alpha\varepsilon \not\leq_{\mathsf{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} * \mathsf{WF}$

$\mathsf{OG} \mapsto \alpha \not\leq_{\mathsf{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$

A slightly bigger part of the zoo

$$
\begin{tikzcd}
 & \widehat{\mathsf{WF}} \equiv_{\mathrm{W}} \mathsf{OG} \mapsto \alpha\varepsilon\mathsf{f} \arrow[dl] \arrow[d] \arrow[ddr] & \\
\mathsf{OG} \mapsto \alpha\varepsilon \arrow[d] \arrow[drr] & \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \arrow[d] & \\
\mathsf{OG} \mapsto \alpha \arrow[d] & \lim \equiv_{\mathrm{W}} \widehat{\mathsf{LPO}} \arrow[dl] \arrow[d] & \mathsf{WF} \equiv_{\mathrm{W}} \mathsf{OG} \mapsto \varepsilon \arrow[dd] \\
\mathsf{OG} \mapsto \alpha 0 \arrow[dr] & \mathsf{C}_{\mathbb{N}} \arrow[d] \arrow[dr] & \\
 & \mathsf{LPO}^* \equiv_{\mathrm{W}} {}^1\mathsf{OG} \mapsto \alpha \equiv_{\mathrm{W}} {}^1\mathsf{OG} \mapsto \alpha 0 \arrow[d] & \lim_2 \arrow[dl] \\
 & \mathsf{Id}_{\mathbb{N}^{\mathbb{N}}} &
\end{tikzcd}
$$

Thank You!