A uniqueness condition for composition analyses

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We will always assume $A < \mathcal{U}$ is a small subset of parameters and that $A = \mathrm{acl}(A)$.

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For any finite rank stationary type $p \in S(A)$, we can break it down into a collection of minimal types.

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Definition

A **semi-minimal analysis** for p is a sequence of A-definable maps

$$p = p_{n+1} \xrightarrow{f_n} p_n \rightarrow \cdots \xrightarrow{f_1} p_1 \xrightarrow{f_0} \bullet$$

such that:

1. $U(p_{i+1}) > U(p_i)$ for all $i = 0, ..., n$, and
2. For all $i = 0, \ldots, n$, every fibre is almost internal to a minimal type.

Definition

A stationary type $p \in S(A)$ is **almost internal** to a minimal type $r \in S(B)$ if there exists $C \supseteq A \cup B$, $a \models p|_C$ and $c_1, ..., c_n \models r|_C$ such that $a \in \mathrm{acl}(Cc_1, ..., c_n)$.

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Are these minimal types "unique"? Do they depend on the analysis?

Fibrations

Let $p, q \in S(A)$ stationary and $f : p \to q$ be an A-definable map.

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- $f : p \to q$ is a **fibration** if every (equivalently some) fibre of f is **stationary** i.e. for every (some) $a \models p$ there is a unique non-forking extension of $\mathrm{tp}(a/Af(a))$ to $\mathrm{tp}(a/\operatorname{acl}(Af(a)))$.

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Definition

Let $p \in S(A), q \in S(B)$ be stationary. We say p is **orthogonal** to q denoted $p \perp q$ if for all $C \supseteq A \cup B$, $a \models p|_C$, $b \models q|_C$, $a \mathop{\smile\!\!\!\!\!\!\mid}_C b$.

Example

Let $p, q \in S(A)$ with $p \perp q$. Recall, $p \otimes q = \mathrm{tp}(ab/A)$ for any $a \models p$ and $b \models q$.

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Let $p, q \in S(A)$ with $p \perp q$. Recall, $p \otimes q = \mathrm{tp}(ab/A)$ for any $a \models p$ and $b \models q$.
The projection map $\pi_p : p \otimes q \to p$ is a fibration and for $(a, b) \models p \otimes q$, the fibre is $q|_a$.

No proper fibrations

Let $p, q \in S(A)$ stationary and $f : p \to q$ be a fibration over A.

Definition

An A-definable map $f : p \to q$ is a **fibration** if every fibre of f is stationary.

Definition

- A fibration $f : p \to q$ is **finite-to-one** if for every $a \models p$, $a \in \mathrm{acl}(Af(a))$.
- A type $p \in S(A)$ admits **no proper fibrations** if and only if for every fibration over A $f : p \to q$ either q is algebraic or f is finite-to-one.

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Example

Every minimal type admits no proper fibrations.
If $U(p) = 1$, $f : p \to q$ is a fibration then either $U(q) = 0$ in which case q is algebraic or $U(q) = 1$. In this case p and q are interalgebraic over A.

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Not every type admitting no proper fibrations is minimal!

No proper fibrations and internality

Let $p, q \in S(A)$ stationary and $f : p \to q$ be a fibration over A.

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- A type p admits **no proper fibrations** if and only if for every fibration $f : p \to q$ either q is algebraic or f is finite-to-one.

Fact

[2, Proposition 2.3] Suppose $p \in S(A)$ is a stationary non-algebraic type of finite U-rank that admits no proper fibrations. Then p is almost internal to a minimal type $r \in (B)$.

Indecomposable fibrations

Let $p, q \in S(A)$ and $f : p \to q$ be a fibration over A with $U(p) > U(q)$.

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A fibration $f : p \to q$ is **indecomposable** if whenever we have fibrations $g : p \to r$, $h : r \to q$ are fibrations with $r \in S(A)$ and $f = h \circ g$, either g or h is finite-to-one.

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Lemma

A fibration $f : p \to q$ is indecomposable if and only if for some (any) $a \models p$, the fibre $\mathrm{tp}(a/f(a)A)$ admits no proper fibrations.

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So f indecomposable $\implies$ each fibre is almost internal to a minimal type.

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- For all $b \models s$, $r|_b$ is minimal $\implies$ π_s is **indecomposable**
- Fix $c \in A$. The map $f_s : s \to \bullet$ given by $f_s(b) = c$ for all $b \models q$. f_s is a fibration with a single fibre $\mathrm{tp}(b/cA) = \mathrm{tp}(b/A) = s$

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Composition analysis

Definition

Let $p \in S(A)$ be a stationary type of finite $U-$rank. A **composition analysis** for p is a sequence of fibrations over A

$$p = p_{n+1} \xrightarrow{f_n} p_n \to \cdots \xrightarrow{f_1} p_1 \xrightarrow{f_0} \bullet$$

such that

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Every stationary type of finite U-rank has a composition analysis.

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- Let $a, a' \models p$ be distinct and $f : p \to q$ be a fibration over A.
- In general, it is possible the fibre $\text{tp}(a/f(a)A)$ is almost internal to a minimal $r \in S(B)$ and $\text{tp}(a'/f(a')A)$ is almost internal to a minimal $r' \in S(B')$ with $B \neq B'$.

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- So a minimal type appearing in a composition analysis might depending on the choice of a fibre!
- Also, is dependent on the choice of r to witness that $\text{tp}(a/f(a)A)$ is almost internal to a minimal type!
- But we can get uniqueness up to non-orthogonality of families of minimal types.

What minimal types appear in a composition analysis?

Definition

- Let $B \supseteq A$, $r \in S(B)$ be minimal. By an$A$-**conjugate** of r we mean a type r^τ, which is obtained by applying some $\tau \in \mathrm{Aut}_A(\mathcal{U})$ to the formulas in r.

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- A family of types $\mathcal{R}$ is A-**invariant** if for every $\tau \in \mathrm{Aut}_A(\mathcal{U})$, whenever $r \in \mathcal{R}$, then $r^\tau \in \mathcal{R}$.

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- A family of types $\mathcal{R}$ is A-**invariant** if for every $\tau \in \mathrm{Aut}_A(\mathcal{U})$, whenever $r \in \mathcal{R}$, then $r^\tau \in \mathcal{R}$.
- [1, Lemma 5.4] A type $p \in S(A)$ is **almost internal** to an A-invariant family of A-conjugates of a minimal type $r \in S(B)$ if and only if p is almost r-internal.

What minimal types appear in a composition analysis?

- Let $f : p \to q$ be an indecomposable fibration. Fix $a \models p$. We know the fibre $\mathrm{tp}(a/f(a)A)$ is almost internal to some minimal type $r_{f(a)} \in S(B_{f(a)})$ with $B_{f(a)} \supseteq f(a)A$.

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- We denote $\min(f)$ to be the set of all A-conjugates of $r_{f(a)}$, i.e.

$$\min(f) = \{r^{\tau}_{f(a)} \mid \tau \in \mathrm{Aut}_A(\mathcal{U})\}.$$

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Uniqueness condition

Theorem

Let $p \in S(A)$ be a stationary type of finite U-rank. Let

$$p \xrightarrow{f_n} p_n \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} p_1 \xrightarrow{f_0} \bullet$$

be a composition analysis for p. Then for any other composition analysis,

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- Up to non-orthogonality, these agree and $\min(r \otimes s) = \{r\} \cup \{s\}$.

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