# Hyperfiniteness, Borel Asymptotic Dimension, and Complexity

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# Motivation

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**SUB-SUB-GOAL (this talk!):** Understand the complexity of "strongly" hyperfinite equivalence relations.

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**Examples**

1. Orbit equivalence relations of Borel actions of countable discrete groups.
2. Connectedness relations of locally countable Borel graphs.

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A CBER is *smooth* if it is Borel reducible to $=_{\mathbb{R}}$.

# Hyperfinite CBERs

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**Example**

The *eventual equality relation* $E_0$ on $\{0,1\}^{\mathbb{N}}$:

$$x \, E_0 \, y \iff \exists n_0 \forall n \geq n_0 (x(n) = y(n)).$$

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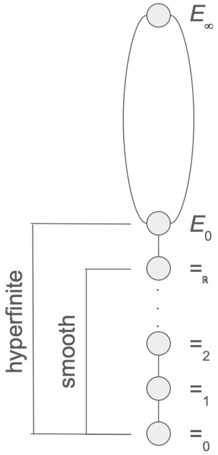
**Theorem: Glimm–Effros Dichotomy**

[Harrington–Kechris–Louveau] Let $E$ be a CBER. If $E$ is not smooth, then $E_0 \leq_B E$.

# Borel Reducibility Hierarchy



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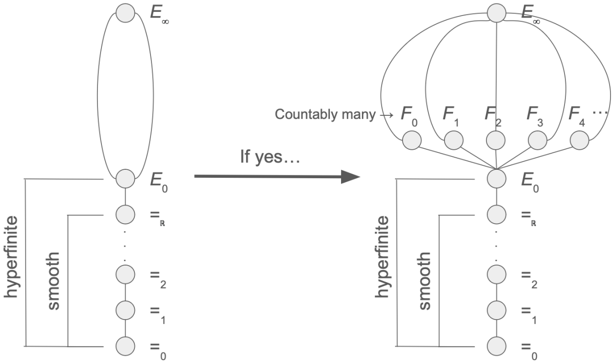

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[Dougherty–Jackson–Kechris] Is hyperfiniteness $\mathbf{\Sigma}^1_2$-complete?

If yes, then no dichotomy theorem for hyperfiniteness is possible.

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- The generic continuous action of Cantor space [Iyer–Shinko].

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[Weiss] Is a Borel action of a countable amenable group hyperfinite?

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- There are acyclic Borel hyperfinite graphs of maximum degree $d$ with no Borel $d$-colorings [Conley–Jackson–Marks–Seward–Tucker-Drob '20].
- Any Borel graph of maximum degree $d \geq 3$ not containing $K_d$ whose Borel asymptotic dimension is finite is Borel $d$-colorable [Bernshteyn–Weilacher].

# Borel Asymptotic Dimension: Definition

Intuitively: A graph $\mathcal{G}$ has Borel asymptotic dimension $s$ if, for each positive $r$, $V(\mathcal{G})$ can be broken into uniformly bounded Borel pieces such that the $r$-neighborhood of each vertex meets only $s + 1$ pieces.

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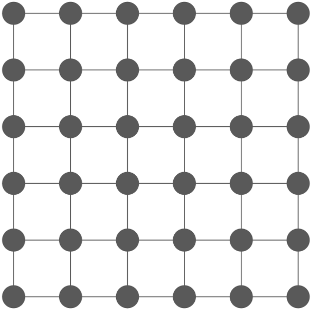

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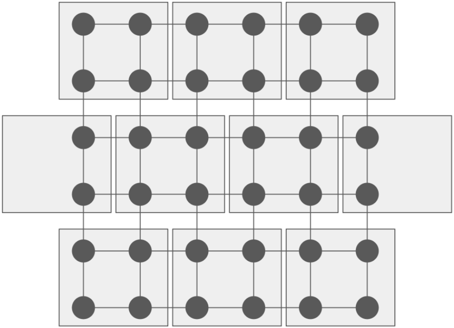

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*Proof sketch:*

1. Reduce the *geometric* question of whether a graph has finite Borel asymptotic dimension to a *purely combinatorial* question on a special class of graphs.
2. Apply the general condition for $\mathbf{\Sigma}^1_2$-completeness due to [Todorčević–Vidnyánszky] and [Frisch–Shinko–Vidnyánszky].

Thank you!