Ramsey spaces and their ultrafilters

Natasha Dobrinen

University of Notre Dame

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Ramsey ultrafilters

An ultrafilter $\mathcal{U}$ on ω is Ramsey iff any of the following hold:

1. For each coloring $f : [\omega]^2 \to 2$, there is an $X \in \mathcal{U}$ such that $f \restriction [X]^2$ is constant. $\mathcal{U} \to (\mathcal{U})^2$
2. For each $n, r \geq 1$ and each coloring $f : [\omega]^n \to r$, there is an $X \in \mathcal{U}$ such that $f \restriction [X]^n$ is constant. $\mathcal{U} \to (\mathcal{U})^n_r$
3. For each $f : \omega \to \omega$ there is an $X \in \mathcal{U}$ such that $f \restriction X$ is either 1-1 or constant. (selective)

Ramsey ultrafilters are constructed using $\mathcal{P}(\omega)/\mathrm{Fin}$ or $([\omega]^\omega, \subseteq^*)$ and CH, MA, $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$, forcing, ...

Special Properties of Ramsey ultrafilters

I. Rudin-Keisler minimal

II. Tukey idempotent

III. Tukey minimal

IV. Barren extension of $L(\mathbb{R})$

V. Complete combinatorics

VI. Preserved by Sacks forcing

Special Properties of Ramsey Ultrafilters

I. Rudin-Keisler minimal (Blass)

$\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U}$ iff $\exists f : \omega \to \omega$ such that $\{f(U) : U \in \mathcal{U}\}$ generates $\mathcal{V}$.
RK-minimality follows from selectivity.

II. Tukey idempotent (Dobrinen–Todorcevic)

III. Tukey minimal (Todorcevic)

$\mathcal{V} \leq_{\mathrm{T}} \mathcal{U}$ iff $\exists f : \mathcal{U} \to \mathcal{V}$ which takes each cofinal subset of $(\mathcal{U}, \supseteq)$ to a cofinal subset of $(\mathcal{V}, \supseteq)$. T-minimality follows from Tukey idempotency, continuous cofinal maps, and Pudlák–Rödl Thm.

IV. Barren extension of $L(\mathbb{R})$ (Henle–Mathias–Woodin)

$L(\mathbb{R})[\mathcal{U}]$ has the same sets of ordinals and same strong partition cardinals as $L(\mathbb{R})$. Proof uses the partition relation $\omega \to (\omega)^\omega$ in $L(\mathbb{R})$.

Special Properties of Ramsey Ultrafilters

V. Complete combinatorics

Any Ramsey ultrafilter in $V[G]$ is $([\omega]^\omega, \subseteq^*)$-generic over $\mathrm{HOD}(\mathbb{R})^{V[G]}$, where $V[G]$ is the Lévy collapse of a Mahlo cardinal to $\aleph_1$. (Blass, building on Mathias)

If V has a supercompact cardinal, then any Ramsey ultrafilter in V is $([\omega]^\omega, \subseteq^*)$-generic over $L(\mathbb{R})$ (Todorcevic, building on Shelah–Woodin)

Proof uses: "All definable sets are $\mathcal{U}$-Ramsey".

VI. Preserved by Sacks forcing (Baumgartner–Laver)

Any Ramsey ultrafilter in V generates a Ramsey ultrafilter after iterated Sacks forcing.

Proof uses: Ellentuck space can be parametrized by Cantor space.

How special are Ramsey ultrafilters?

Do other ultrafilters have similar properties?

What is really responsible for these special properties?

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Topological Ramsey Spaces

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A triple $(\mathcal{R}, \leq, r = (r_n)_{n<\omega})$ is a **topological Ramsey space (TRS)** iff $\mathcal{R}$ is a topological space in which every subset of $\mathcal{R}$ with the property of Baire is Ramsey, and every meager subset of $\mathcal{R}$ is Ramsey null.

Members $A \in \mathcal{R}$ are infinite sequences, and $r_n(A)$ is the n-th finite restriction of A. $\mathcal{AR}_1 = \{r_1(A) : A \in \mathcal{R}\}$.

$\mathcal{X} \subseteq \mathcal{R}$ is **Ramsey** iff $\forall A, C \in \mathcal{R}$ and $c = r_n(C)$ for some n,

$$\forall \emptyset \neq [c, A] \ \exists B \in [c, A] \ ([c, B] \subseteq \mathcal{X} \ \vee \ [c, B] \cap \mathcal{X} = \emptyset).$$

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Any generic filter $\mathcal{G}$ on $\mathcal{R}$ induces an ultrafilter $\mathcal{U}_{\mathcal{R}}$ on base $\mathcal{AR}_1$.

Theorem (Navarro Flores)

If $(\mathcal{R}, \leq)$ does not add reals, then it forces a Ramsey ultrafilter.

Examples of Ultrafilters from TRS's

weakly Ramsey: $\mathcal{U}_1 \to (\mathcal{U}_1)^2_{\ell,2}$

Laflamme ultrafilters: $\mathcal{U}_k \to (\mathcal{U}_k)^2_{\ell,k+1}$

Blass n-square forcing: $\mathcal{U} \to (\mathcal{U})^2_{\ell,5}$

Baumgartner–Taylor *k*-arrow ultrafilters: $\mathcal{U} \to (\mathcal{U}, k)^2$

Forced by $\mathcal{P}(\omega \times \omega)/\mathrm{Fin} \otimes \mathrm{Fin}$: $\mathcal{U} \to (\mathcal{U})^2_{\ell,4}$

Forced by $\mathcal{P}(\omega^\alpha)/\mathrm{Fin}^{\otimes\alpha}$, $1 \leq \alpha < \omega_1$

stable ordered union ultrafilters – see Tan Özalp's talk on Wednesday

And many others.

For references, see "Topological Ramsey spaces dense in forcings," by Dobrinen, in *Structure and Randomness in Computability and Set Theory*, World Scientific (2020), 3–58.

V. Selective Coideals and Complete Combinatorics

Given a TRS $(\mathcal{R}, \leq, r)$, a coideal $\mathcal{C} \subseteq \mathcal{R}$ is **selective** iff $\forall A \in \mathcal{U}$ and $\forall (A_b)_{b \in \mathcal{AR}|A}$ there is a $C \in \mathcal{C}$ that diagonalizes $(A_b)_{b \in \mathcal{AR}|C}$.

Forcing with $(\mathcal{R}, \leq_M^*)$ adds a selective coideal on $\mathcal{R}$, whenever $\leq_M^*$ is σ-closed.

Theorem (Di Prisco–Mijares–Nieto)

In the presence of a supercompact cardinal, every selective coideal on $\mathcal{R}$ is generic for $(\mathcal{R}, \leq_M^*)$ over $L(\mathbb{R})$.

$\leq_M^*$ denotes Mijares' weakening of $\leq$.

RK and Tukey Reductions

$\mathcal{V}$ is **Rudin-Keisler reducible** to $\mathcal{U}$ ($\mathcal{V} \leq_{RK} \mathcal{U}$) if there is a map $f : \omega \to \omega$ such that $\{f(U) : U \in \mathcal{U}\}$ generates $\mathcal{V}$.

$\mathcal{V}$ is **Tukey reducible** to $\mathcal{U}$ ($\mathcal{V} \leq_T \mathcal{U}$) if there is a map $f : \mathcal{U} \to \mathcal{V}$ such that each f-image of a filter base for $\mathcal{U}$ is a filter base for $\mathcal{V}$.

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Tukey types of ultrafilters coarsen the Rudin-Keisler types:

$$\mathcal{V} \leq_{RK} \mathcal{U} \Longrightarrow \mathcal{V} \leq_T \mathcal{U}.$$

Canonical Cofinal Maps

- If $\mathcal{U}$ has canonical cofinal maps, then $|[\mathcal{U}]_T| \leq \mathfrak{c}$.

Theorems.

1. All p-points have continuous Tukey reductions. [DT1]
2. All ultrafilters Tukey reducible to a p-point have continuous Tukey reductions. [D3]
3. All ultrafilters Tukey reducible to a Fubini iterate of p-points have finitely generated Tukey reductions. [D3]
4. All ultrafilters generated by TRS's investigated so far have finitely generated Tukey reductions. [DMT], [D2]

Open Problem. Does $(\mathcal{U}, \supseteq) <_T ([\mathfrak{c}]^{<\omega}, \subseteq)$ imply $\mathcal{U}$ has definable Tukey reductions?

- If $\mathcal{U}$ is a rapid p-point, then $\mathcal{U} \cdot \mathcal{U} \equiv_T \mathcal{U}$. [DT1]

- For every nonprincipal ultrafilter $\mathcal{U}$, $\mathcal{U} \cdot \mathcal{U} \cdot \mathcal{U} \equiv_T \mathcal{U} \cdot \mathcal{U}$. [Milovich 2]

- If an ultrafilter $\mathcal{U}$ has the I-p.i.p. for some ideal $I \subseteq \mathcal{U}^*$, then $\mathcal{U} \cdot \mathcal{U} \equiv_T \mathcal{U}$. [BD]

  Examples: Milliken–Taylor ultrafilters on FIN and ultrafilters forced by $\mathcal{P}(\omega^{\otimes\alpha})/\mathrm{Fin}^{\otimes\alpha}$, $\alpha < \omega_1$.

- General TRS ultrafilters - ongoing work of [BDÖ].

  Example: Milliken–Taylor ultrafilters on FIN_k.

For $\mathcal{U}$ an ultrafilter and $I \subseteq \mathcal{U}^*$ an ideal: $\mathcal{U}$ has the ***I*-pseudo intersection property** iff $\forall \langle A_n : n < \omega \rangle \subseteq \mathcal{U}\ \exists A \in \mathcal{U}$ such that for each n, $A_n \setminus A \in I$.

I. & III. Initial Structures

A downwards closed collection of Tukey types of nonprincipal ultrafilters $(C, \leq_T)$ is an **initial Tukey structure**.

Thm. [Todorcevic in [RT]] If $\mathcal{U}$ is a Ramsey ultrafilter and $\mathcal{V} \leq_T \mathcal{U}$, then either $\mathcal{V}$ is isomorphic to a Fubini iterate of $\mathcal{U}$, or else $\mathcal{V}$ is principal. Thus, the Tukey type of a Ramsey ultrafilter is minimal.

Todorcevic's proof makes essential use of
- continuous Tukey reductions
- Tukey idempotency of Ramsey ultrafilters
- Ramsey-classification theorem of Pudlák and Rödl.

Methodology for Initial Structures

The methodology employed for finding initial Tukey and Rudin-Keisler structures involve

1. Possibly construct a new topological Ramsey space forcing equivalent to the forcing generating a given ultrafilter.
2. Prove canonical equivalence relations on fronts on the TRS.
3. Prove that cofinal maps are enough like continuous maps (finitary).
4. Use Tukey idempotency and commutativity of the relevant ultrafilters.
5. Applying these to decode (a) the initial RK structure, (b) the RK types inside each Tukey type below $\mathcal{U}$, (c) the initial Tukey structure.

Initial Tukey and Rudin-Keisler structures

All ultrafilters on slide 8 (and more) have well-understood initial RK and initial Tukey structures.

See Özalp's talk on Wednesday answering a question of Dobrinen and Todorcevic about the initial Tukey structure below a stable ordered union ultrafilter.

Relevant references are [DT2,3], [D2,3,5], [DMT], [Ö].

IV. Barren Extensions

Theorem (Dobrinen/Hathaway)

Suppose $\mathcal{U}$ is forced by $(\mathcal{R}, r, \leq)$ satisfying Todorcevic's Axioms A.1–A.4 and a certain 'Left-Right' Axiom.

1. *Then $L(\mathbb{R})$ and $L(\mathbb{R})[\mathcal{U}]$ have the same sets of ordinals.*
2. *If moreover there is a σ-closed coursening $\leq^*$ which has countable $=^*$-equivalence classes and $(\mathcal{R}, \leq)$ is forcing equivalent to $(\mathcal{R}, \leq^*)$, then $L(\mathbb{R})$ and $L(\mathbb{R})[\mathcal{U}]$ have the same strong partition cardinals.*

This generalizes the theorem of Henle–Mathias–Woodin showing that Ramsey ultrafilters have these properties.

VI. Ultrafilters Preserved by Sacks Forcing

Theorem (Baumgartner–Laver)

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Theorem (Zheng1,2,3)

Suppose $(\mathcal{R}, \leq, r)$ is a closed triple satisfying an additional Axiom (L4). Then its forced ultrafilter is preserved by countable support product Sacks forcing.

Examples: stable ordered union ultrafilters and all ultrafilters in [DT1,2] and [D2].

Related works by Yiparaki and by Chodounský–Guzmán–Hrušak on preservation and HL ideals.

VI. Ultrafilters Preserved by Sacks Forcing

Theorem (D–Zhang)

Suppose $(\mathcal{R}, \leq, r)$ is a closed triple satisfying an additional Axiom (L4$^-$) and the Independent Finite Pigeonhole. Then its forced ultrafilter is preserved by product and by iterated Sacks forcing.

Examples: FIN_k ultrafilters, and all ultrafilters in [DMT], $\mathcal{P}(\omega^\alpha)/\mathrm{Fin}^{\otimes\alpha}$ ultrafilters, and generally those from TRS's with Independent Sequencing.

Parametrized Ramsey Space Theorems

Theorem (Zheng)

Let $\mathcal{R}$ be a TRS satifsying (L4). For every finite Souslin-measurable coloring of $\mathcal{R} \times \mathbb{R}^\omega$, there are $A \in \mathcal{R}$ and $(P_i)_{i<\omega}$ perfect sets of reals such that $[\emptyset, A] \times \prod_{i<\omega} P_i$ is monochromatic.

(L4) Let $A \in \mathcal{R}$ and $a \in \mathcal{AR}$. Let $\{O_b : b \in r_{|a|+1|}[a, A]\}$ be a family of open subsets of $(2^\omega)^\omega$. Then $\exists\ B \in [\text{depth}_A(a), A]$, $a \in \mathbb{S}_\omega$, and a clopen subset $G \subseteq [q]$ such that $\mathcal{O}_b \cap [q] = G$, $\forall b \in r_{|a|+1}[a, B]$.

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Theorem (D–Zhang)

If $\mathcal{R}$ satisfies (L4$^-$) and the Independent Finite Pigeonhole Property, then $\mathcal{R}$ satisfies (L4).

(L4$^-$) Given $A \in \mathcal{R}$, $d < \omega$, and $N \in [\omega]^\omega$, there is a $B \in [d, A]$ such that for each $k \geq d$, $\mathrm{depth}_A(r_k(B)) \in N$.

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