Extending affine subspaces in higher dimensions

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Problems in Geometric Measure Theory

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- Usually, by size we mean measure or some notion of dimension (e.g. Hausdorff, box counting, packing, Assouad...).
- More precisely, what can we say about sets that we know have a certain geometric property, or that are obtained by some natural geometric operation on a set with known size?

Fractal dimension

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In general, $\dim_H(E) \leq \dim_P(E)$

Effective dimension

Definition

The effective Hausdorff dimension of a point $x \in \mathbb{R}^n$ relative to an oracle $A \subseteq \mathbb{N}$ is given by

$$\dim^A(x) = \liminf_{r \to \infty} \frac{K_r^A(x)}{r}$$

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The effective packing dimension of a point $x \in \mathbb{R}^n$ relative to an oracle $A \subseteq \mathbb{N}$ is given by

$$\mathrm{Dim}^A(x) = \limsup_{r\to\infty} \frac{K_r^A(x)}{r}$$

The point-to-set principle

Effective dimension is directly related to classical dimension through the following "point-to-set" principle(s):

Theorem (J. Lutz and N. Lutz, 2015)

For all $E \subset \mathbb{R}^n$,

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x)$$

and

$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \mathrm{Dim}^A(x)$$

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k-planes and the Grassmanian

- Let $n \geq 2$ and $1 \leq k \leq n-1$ be given. We denote the set of k-planes in $\mathbb{R}^n$ passing through the origin (the Grassmanian) by $\mathcal{G}(n, k)$.

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- There are a number of equivalent metrics we can define on the affine Grassmanian, meaning we can talk about the Hausdorff and packing dimension of subsets of $\mathcal{A}(n,k)$.

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The Furstenberg set problem

Let $E \subseteq \mathbb{R}^2$ be such that there is a Hausdorff dimension t set of lines, each intersecting E in a set of Hausdorff dimension s. Then

$$\dim_H(E) \geq \min\{s + t, s + 1, \frac{3s + t}{2}\}$$

k-plane extension

Another sub-question: Suppose the subsets under consideration are all large, e.g. the whole k-plane, nonempty interior, positive measure, Hausdorff dimension k. Can their *exact* size affect the size of the union?

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More precisely, given any $E \subseteq \mathbb{R}^n$, let $\mathcal{I}_k(E) \subseteq \mathcal{A}(n,k)$ index the k-planes intersecting E in a set of Hausdorff dimension k. Define

$$\mathcal{P}_k(E) = E \cup \bigcup_{I \in \mathcal{I}_k(E)} I$$

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Conjecture

Let $E \subseteq \mathbb{R}^n$. For all $1 \leq k \leq n-1$,

$$\dim_H(E) = \dim_H(\mathcal{P}_k(E)) \quad \text{and} \quad \dim_P(E) = \dim_P(\mathcal{P}_k(E))$$

Results for lines

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In $\mathbb{R}^3$:

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In higher dimensions:

- (Bushling and F., 2024): Extending a collection of line segments to full lines does not increase the packing dimension to more than $2\dim_P(E) - 1$

New results

In general, we have the following bound for the extension of k-planes:

Theorem (F., 2025)

Let $E \subseteq \mathbb{R}^n$ be a union of open subsets of k-planes. Then, if F_k is the set formed by extending each of these subsets,

$$\dim_P(F_k) \leq 2\dim_P(E) - k$$

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Moreover, we have an improved result for hyperplanes:

Theorem (F., 2025)

Let $E \subseteq \mathbb{R}^n$. Then

$$\dim_P(\mathcal{P}_{n-1}(E)) = \dim_P(E)$$

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Effectivization of k-plane extension

- Using the point-to-set principle

$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \mathrm{Dim}^A(x) = \min_{A \subseteq \mathbb{N}} \sup_{I \in \mathcal{I}} \sup_{x \in E \cap I} \mathrm{Dim}^A(x)$$

$$\dim_P(F_k) = \min_{A \subseteq \mathbb{N}} \sup_{x \in F_k} \mathrm{Dim}^A(x) = \min_{A \subseteq \mathbb{N}} \sup_{I \in \mathcal{I}} \sup_{x \in I} \mathrm{Dim}^A(x)$$

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- So it suffices to show that for every $A \subseteq \mathbb{N}$, $I \in \mathcal{A}(n,k)$, and S an open subset of I,

$$\sup_{x\in I} \mathrm{Dim}^A(x) \leq 2 \sup_{x\in S} \mathrm{Dim}^A(x) - k$$

Choosing points

- Using the properties of the limit superior this follows if for every $x \in I$, there exist $y, z \in S$ such that,

$$K_r^A(x) \leq K_r^A(y) + K_r^A(z) - kr$$

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- Choose y so that it's first k coordinates are random relative to x and A; call this vector $\hat{y}$. We may also assume the line containing x and y intersects S in a set of positive measure.
- Hence, we may choose z on this line such that one of its first k coordinates is random relative to x, y, and A ; call this coordinate $\hat{z}$.

Completing the proof

Main idea: There exists some $t \in \mathbb{R}$ such that $x = y + t(z - y)$. Hence a precision r estimate of y, z, and t is enough to compute a (nearly) precision r estimate of x and other *unrelated* information.

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On one hand,

$$\begin{aligned} K_r^A(x, \hat{y}, \hat{z}) &\leq K_r^A(y, z, t) \\ &\leq K_r^A(y) + K_r^A(z) + K_r^A(t) \\ &\leq K_r^A(y) + K_r^A(z) + r \end{aligned}$$

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On the other,

$$\begin{aligned} K_r^A(x, \hat{y}, \hat{z}) &= K_r^A(x) + K_r^A(\hat{y}, \hat{z}) \\ &= K_r^A(x) + K_r^A(\hat{y}) + K_r^A(\hat{z}) \\ &= K_r^A(x) + kr + r \end{aligned}$$

Sketch of the proof for hyperplanes I

Geometric lemma

Let $n \geq 2$ and a hyperplane $(a, b) = (a_1, ...a_{n-1}, b)$ be given and assume $(u, v) = (u_1, ...u_{n-1}, v)$ is a hyperplane such that $a_1x_1 + ... + a_{n-1}x_{n-1} + b = u_1x_1 + ... + u_{n-1}x_{n-1} + v$. Let $t \leq r$ be the largest precision up to which (a, b) and (u, v) agree. Then for every oracle A,

$$K_r^A(u, v) \geq K_t^A(a, b) + K_{r-t,r}^A(x_1, ...x_{n-1}|a, b) - (n-2)(r-t) - O(\log r)$$

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Idea: Two intersecting hyperplanes determine an $(n-2)$-plane (with a possible loss of precision). $(n-2)(r-t)$ bits of extra information determine any specific point in this $(n-2)$-plane.

Sketch of the proof for hyperplanes II

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- Applying the geometric lemma to a nearly random $x_1, ...x_n$ (which we have the freedom to pick) gives the second condition.
- A short argument shows that the resulting lower bound on the complexity of the point nearly matches a natural upper bound at every precision, completing the proof.

Thank you!