Properties of Selector Proofs

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Notation

Let $S \subseteq T$ be consistent, elementary-recursively axiomatisable, and contain $I\Sigma_0 + Exp$. We assume that S is Σ_1-sound.

- $x{:}_T y$ will be a standard proof predicate
- $\Box_T y$ will be the corresponding provability predicate for T: $\exists x (x{:}_T y)$
- We shall omit Gödel codes and write $x{:}_T A$, $\Box_T A(x)$ etc.

Our results are all relative to a fixed proof predicate for T. The situation changes when the proof predicate is allowed to vary and non-standard proof predicates are admitted (Santos, Sieg and Kahle, 2023).

Serial Properties

- A serial property is an effectively enumerated sequence $\{F_n\}$ of formulas
- We will focus on the case where $\{F_n\}$ consists of all instances of a single formula

What does it mean to say that T proves a serial property?

- Instance provability is not enough
- (Assuming that ZF is consistent) $PA \vdash \neg \underline{n}{:}_{ZF} \bot$ for each n

Selector Proofs

Definition

A formula $F(\vec{x})$ is selector provable in T (over the base theory S) if there is an S-recursive function $s(\vec{x})$ such that

$$S \vdash \forall \vec{x}(s(\vec{x}){:}_T F(\vec{x}))$$

From such a selector proof, it follows by explicit reflection that $T \vdash F(\underline{\vec{n}})$ for each $\vec{n}$ (without metamathematical assumptions on T).

Selector-Provable Formulas

Various formulas are selector provable, but not provable.

Theorem (Artemov)

The formula $\neg(x{:}_{PA}\bot)$ is selector provable in PA. There is an elementary-recursive function $s(x)$ such that

$$I\Sigma_0 + Exp \vdash \forall x(s(x){:}_{PA}\neg(x{:}_{PA}\bot))$$

Artemov argues his particular function $s(x)$ is a formalisation of a contentual consistency proof for PA. We will not address the matter.

Other useful examples (for PA) include:

- For each formula $A(x)$, the formula $TI(\alpha, A)$ asserting transfinite induction for $A(x)$ up to the ordinal $\alpha < \epsilon_0$
- The formula $\exists y(F_{\epsilon_0}(x) = y)$ asserting the totality of the fast-growing function F_{ϵ_0}.

However, $\exists y(F_{\epsilon_0+1}(x) = y)$ is not selector provable.

Δ_1 formulas

Can we characterise the selector-provable formulas?

Proposition (Ignjatović, Kurahashi, Sinclaire)

Let $F(\vec{x})$ be Δ_1 in S. Then

$$S \vdash \forall \vec{x} \Box_T F(\vec{x}) \leftrightarrow (Con(T) \rightarrow \forall \vec{x} F(\vec{x}))$$

Consistency Statements

Corollary

T selector proves the consistency of W (over S) iff $S \vdash Con(T) \to Con(W)$

In other words, T selector proves the consistency of W iff W is relatively consistent with T.

Corollary (Kurahashi-Sinclaire)

If $W \supseteq T + Con(T)$, then $S \nvdash \forall x \Box_T \neg(x{:}_W \bot))$

Are selector proofs a useful tool in proof theory?

Theorem (Freund and Pakhomov 2020)

Let T be the theory $PA + SRefl_{\Sigma_1}(PA)$. Then there is an elementary recursive function $s(x)$ such that

$$I\Sigma_0 + Exp \vdash \forall x(s(x){:}_{PA}\neg(x{:}_T\bot))$$

Thus $I\Sigma_0 + Exp \vdash Con(PA) \rightarrow Con(PA + SRefl_{\Sigma_1}(PA))$

Proof Outline.

- Embed the proof into an infinitary proof system with the ω-rule.
- An instance of $\exists y(F_{\epsilon_0}(x) = y)$ (that depends on the proof) implies that it is not a proof of the empty sequent.
- The role of the selector is to give the proof of the relevant instance of $\exists y(F_{\epsilon_0}(x) = y)$

$\square$

Extensions of *PA*

Corollary

For each $\alpha < \epsilon_0$, there is an elementary recursive function $s(x)$ such that

$$I\Sigma_0 + Exp \vdash \forall x(s(x){:}_{PA}\neg(x{:}_{T_\alpha}\bot))$$

where $T_\alpha = PA + SCon^\alpha(PA)$

- For finite iterates, this can also be established by bounding the length of the Gentzen reduction procedure in terms of F_{ϵ_0}.
- These selector proofs are contentual in Artemov's sense.

Can we extend the characterisation of the selector-provable formulas?

Proposition

Let Γ be a set of formulas containing every Σ_1 formula or every Π_1 formula. There is no set of sentences Δ such that for each formula $F(\vec{x})$ in Γ

$$S \vdash \forall \vec{x} \Box_T F(\vec{x}) \qquad \textit{iff} \qquad S + \Delta \vdash \forall \vec{x} F(\vec{x})$$

Proof sketch for the Σ_1 case.

- The explicit reflection formula $y{:}_T F(\vec{x}) \to F(\vec{x})$ is selector provable in T for each formula F.
- So if the equivalence holds, then $S + \Delta \vdash \mathrm{Refl}_{\Sigma_1}(T)$
- But then $S + \Delta \vdash Con(T + Con(T))$ which is not selector provable.

$\square$

One layer of coding allows for new formulas to become (selector) provable. What about additional layers of coding?

Definition

A formula $F(\vec{x})$ is 2-selector provable in T (over S) if there are S-recursive functions $r(\vec{x})$ and $s(\vec{x})$ such that

$$S \vdash \forall \vec{x}(r(\vec{x}){:}_T(s(\vec{x}){:}_T F(\vec{x})))$$

Can this get us anything new?

No!

Proposition

If a formula is 2-selector provable, then it is selector provable.

This contrasts with the case for iterated provability predicates, where there is a strict hierarchy (Kushida 2020).

The Choice of Base Theory

- While primitive recursive selectors (in other words $S = I\Sigma_1$) seem more than sufficient for ordinary purposes, in principle the full proof-theoretic strength of T is needed in the base theory.
- (Wainer, Schwichtenberg) A function is provably recursive in PA iff it is elementary recursive in F_α for some $\alpha < \epsilon_0$.

Theorem

For each $2 \leq \alpha < \epsilon_0$, there is a ($\Delta_1$ in PA) formula $F(x)$ that is selector provable in PA by a selector elementary recursive in F_α, but not by any selector elementary recursive in F_β for $\beta < \alpha$.

A similar construction works for other theories.

Thank you!

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