# Separating complexity classes of LCL problems on grids

Katalin Berlow, Anton Bernshteyn, Clark Lyons, Felix Weilacher

ASL Meeting, May 2025

# Outline

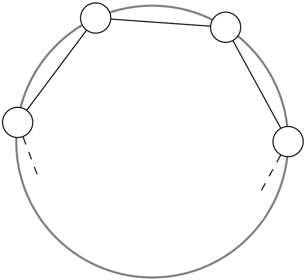
Descriptive Combinatorics

Complexity classes of LCLs

Connections with other areas

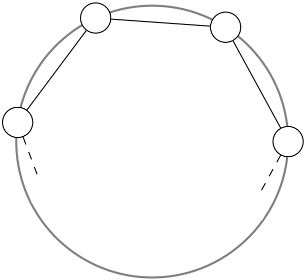
Toast

# Example 1: A coloring problem on the circle

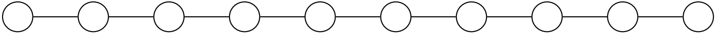


- Consider the graph on $S^1$ where two points are adjacent if they are 1 radian apart. Each connected component is a bi-infinite path.

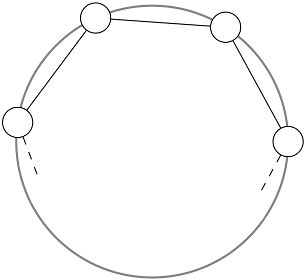
# Example 1: A coloring problem on the circle



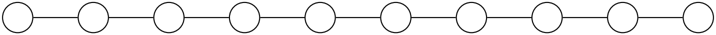
- Consider the graph on $S^1$ where two points are adjacent if they are 1 radian apart. Each connected component is a bi-infinite path.



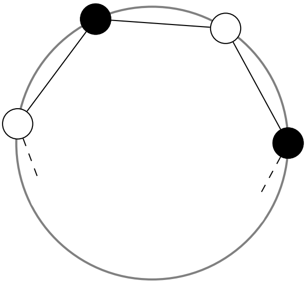
# Example 1: A coloring problem on the circle



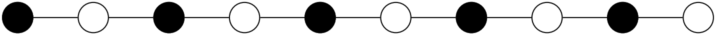
- Consider the graph on $S^1$ where two points are adjacent if they are 1 radian apart. Each connected component is a bi-infinite path.
- Let's try and *properly color* the graph: assign colors to vertices so that adjacent vertices get different colors.



# Example 1: A coloring problem on the circle



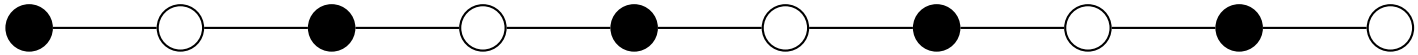
- Consider the graph on $S^1$ where two points are adjacent if they are 1 radian apart. Each connected component is a bi-infinite path.
- Let's try and *properly color* the graph: assign colors to vertices so that adjacent vertices get different colors.
- Working on one component at a time, we see that 2 colors are enough.



# Example 1: A coloring problem on the circle

**Proposition**

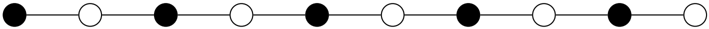
*This graph has no Borel 2-coloring.*



# Example 1: A coloring problem on the circle

**Proposition**

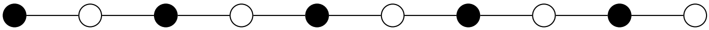
*This graph has no Borel 2-coloring. In fact, it has no Lebesgue or Baire measurable 2-coloring.*



# Example 1: A coloring problem on the circle

**Proposition**

*This graph has no Borel 2-coloring. In fact, it has no Lebesgue or Baire measurable 2-coloring.*

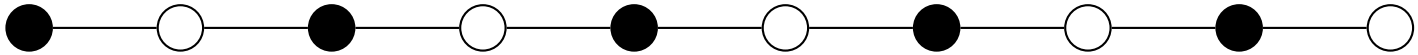


- Let $A$ and $B$ be the sets of black and white points respectively. WLOG, $A$ is nonmeager. Let $I \subseteq S^1$ be an open interval on which $A$ is comeager.

# Example 1: A coloring problem on the circle

**Proposition**

*This graph has no Borel 2-coloring. In fact, it has no Lebesgue or Baire measurable 2-coloring.*

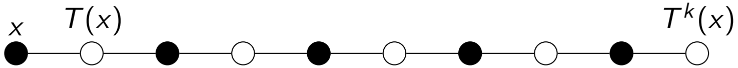


- Let $A$ and $B$ be the sets of black and white points respectively. WLOG, $A$ is nonmeager. Let $I \subseteq S^1$ be an open interval on which $A$ is comeager.
- Let $T : S^1 \to S^1$ be rotation by 1 radian.

# Example 1: A coloring problem on the circle

**Proposition**

*This graph has no Borel 2-coloring. In fact, it has no Lebesgue or Baire measurable 2-coloring.*

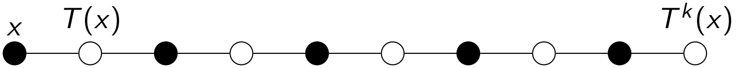



- Let $A$ and $B$ be the sets of black and white points respectively. WLOG, $A$ is nonmeager. Let $I \subseteq S^1$ be an open interval on which $A$ is comeager.
- Let $T : S^1 \to S^1$ be rotation by 1 radian.
- Note $T(A) = B$, $T(B) = A$.

# Example 1: A coloring problem on the circle

**Proposition**

*This graph has no Borel 2-coloring. In fact, it has no Lebesgue or Baire measurable 2-coloring.*




- Let $A$ and $B$ be the sets of black and white points respectively. WLOG, $A$ is nonmeager. Let $I \subseteq S^1$ be an open interval on which $A$ is comeager.
- Let $T : S^1 \to S^1$ be rotation by 1 radian.
- Note $T(A) = B$, $T(B) = A$.
- But by irrationality, there is some odd $k$ with $T^k(I) \cap I \neq \emptyset$. Since $T$ is a homeomorphism, $T^k(A) \cap A$ is comeager in this interval, but $T^k(A) \cap A = B \cap A = \emptyset$.

# Example 1: A coloring problem on the circle

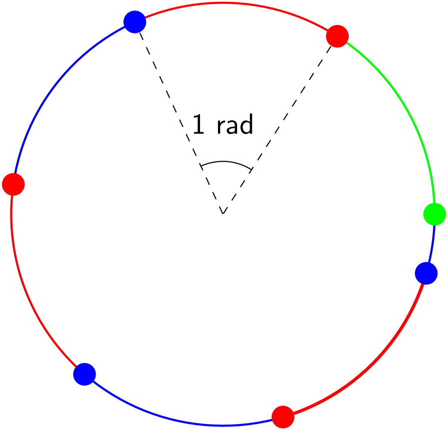
**Proposition (Kechris-Solecki-Todorčević '99)**

*This graph has a Borel 3-coloring.*

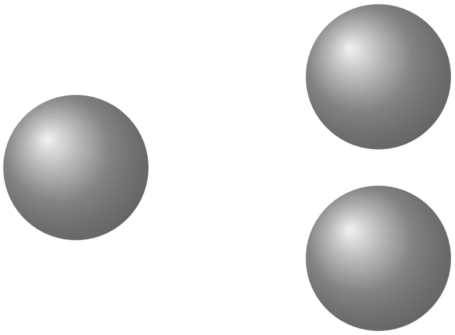
# Example 1: A coloring problem on the circle

**Proposition (Kechris-Solecki-Todorčević '99)**

*This graph has a Borel 3-coloring.*

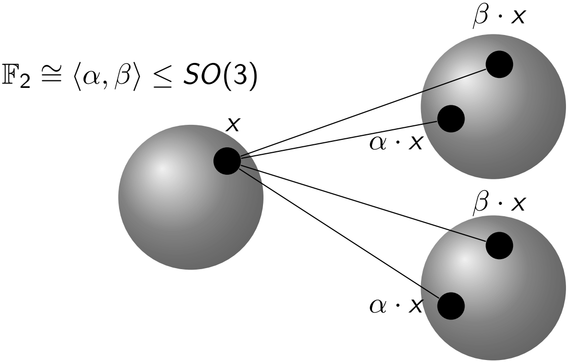


# Example 2: The Banach-Tarski paradox



# Example 2: The Banach-Tarski paradox

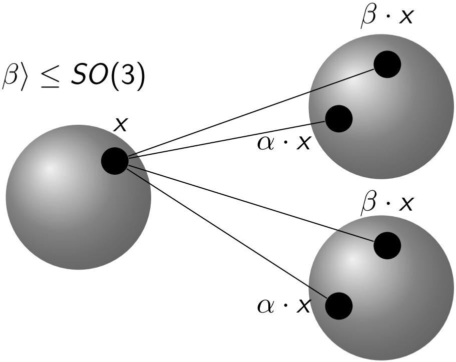
- Consider the bipartite graph on these spheres which connects each point on the left to its images under certain rotations on the right.



# Example 2: The Banach-Tarski paradox

- Consider the bipartite graph on these spheres which connects each point on the left to its images under certain rotations on the right.
- The paradox amounts to the existence of a perfect matching in this graph.

$$\mathbb{F}_2 \cong \langle \alpha, \beta \rangle \leq SO(3)$$



# Example 2: The Banach-Tarski paradox

- Consider the bipartite graph on these spheres which connects each point on the left to its images under certain rotations on the right.
- The paradox amounts to the existence of a perfect matching in this graph.

# Example 2: The Banach-Tarski paradox

- Consider the bipartite graph on these spheres which connects each point on the left to its images under certain rotations on the right.
- The paradox amounts to the existence of a perfect matching in this graph.
- Such a matching cannot be Borel!

# Example 2: The Banach-Tarski paradox

- Consider the bipartite graph on these spheres which connects each point on the left to its images under certain rotations on the right.
- The paradox amounts to the existence of a perfect matching in this graph.
- Such a matching cannot be Borel!

**Theorem (Dougherty-Foreman '94)**

*There is a paradoxical decomposition of the sphere using Baire measurable pieces.*

# Example 2: The Banach-Tarski paradox

- Consider the bipartite graph on these spheres which connects each point on the left to its images under certain rotations on the right.
- The paradox amounts to the existence of a perfect matching in this graph.
- Such a matching cannot be Borel!

**Theorem (Dougherty-Foreman '94)**

*There is a paradoxical decomposition of the sphere using Baire measurable pieces.*
*Alternatively, the graph above has a Borel matching which covers a comeager set of points.*

# Example 3: Tarski's circle squaring problem

**Question (Tarksi 1925)**

*Are a square and disk of the same area in $\mathbb{R}^2$ equidecomposable?*

# Example 3: Tarski's circle squaring problem

**Question (Tarksi 1925)**

*Are a square and disk of the same area in* $\mathbb{R}^2$ *equidecomposable?*

**Theorem (Laczkovich '90)**

*Yes, with translations only.*

# Example 3: Tarski's circle squaring problem

**Question (Tarksi 1925)**

*Are a square and disk of the same area in $\mathbb{R}^2$ equidecomposable?*

**Theorem (Laczkovich '90)**

*Yes, with translations only.*

**Theorem (Grabowski-Máthé-Pikhurko '16)**

*Yes, with Lebesgue and Baire measurable pieces!*

# Example 3: Tarski's circle squaring problem

**Question (Tarksi 1925)**

*Are a square and disk of the same area in $\mathbb{R}^2$ equidecomposable?*

**Theorem (Laczkovich '90)**

*Yes, with translations only.*

**Theorem (Grabowski-Máthé-Pikhurko '16)**

*Yes, with Lebesgue and Baire measurable pieces!*

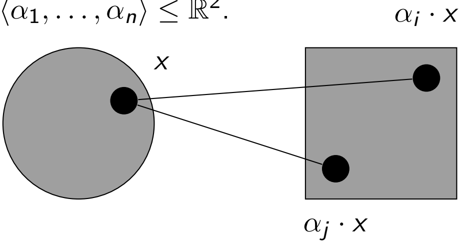
**Theorem (Marks-Unger '17)**

*Yes, with Borel pieces!!*

# Example 3: Tarski's circle squaring problem

- As before, the results amount to finding perfect matchings in graphs generated by certain translations.

$$\mathbb{Z}^n \cong \langle \alpha_1, \ldots, \alpha_n \rangle \leq \mathbb{R}^2.$$



# Outline

Descriptive Combinatorics

Complexity classes of LCLs

Connections with other areas

Toast

# LCLs on groups

We desire an abstract and general framework for discussing these sorts of problems.

# LCLs on groups

We desire an abstract and general framework for discussing these sorts of problems.

**Definition**

Let $\Gamma$ be a countable group. An **LCL** on $\Gamma$ is a triple $\Pi = (\Lambda, \mathcal{W}, \mathcal{A})$, where

- $\Lambda$ is a finite set (**labels**)
- $W \subset \Gamma$ is finite (**window**)
- $\mathcal{A} \subseteq \Lambda^W$ (**allowed configurations**)

# LCLs on groups

We desire an abstract and general framework for discussing these sorts of problems.

**Definition**

Let $\Gamma$ be a countable group. An **LCL** on $\Gamma$ is a triple $\Pi = (\Lambda, \mathcal{W}, \mathcal{A})$, where

- $\Lambda$ is a finite set (**labels**)
- $W \subset \Gamma$ is finite (**window**)
- $\mathcal{A} \subseteq \Lambda^W$ (**allowed configurations**)

- If $\Gamma \curvearrowright X$ is a free action of $\Gamma$ on a set $X$, a $\Pi$-**labeling** of $X$ (also called a **solution** to $\Pi$) is a function $c : X \to \Lambda$ such that for all $x \in X$, the function $W \to \Lambda$ given by $\gamma \mapsto c(\gamma \cdot x)$ is in $\mathcal{A}$.

# Examples

- Example 1: $\mathbb{Z} \curvearrowright S^1$, where $n \in \mathbb{Z}$ acts by rotation by $n$ radians. $\Pi_k = (k, \{0, 1\}, \mathcal{A})$, where $\mathcal{A} = \{c \mid c(0) \neq c(1)\}$.

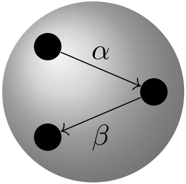
# Examples

- Example 1: $\mathbb{Z} \curvearrowright S^1$, where $n \in \mathbb{Z}$ acts by rotation by $n$ radians. $\Pi_k = (k, \{0,1\}, \mathcal{A})$, where $\mathcal{A} = \{c \mid c(0) \neq c(1)\}$. Proper $k$-colorings of $S^1$ are exactly $\Pi_k$-labelings of $S^1$.

# Examples

- Example 1: $\mathbb{Z} \curvearrowright S^1$, where $n \in \mathbb{Z}$ acts by rotation by $n$ radians. $\Pi_k = (k, \{0,1\}, \mathcal{A})$, where $\mathcal{A} = \{c \mid c(0) \neq c(1)\}$. Proper $k$-colorings of $S^1$ are exactly $\Pi_k$-labelings of $S^1$.
- Example 2: $\mathbb{F}_2 \curvearrowright S^2$ (ignore the non-free part). Label each point with the rotation it will be moved by.

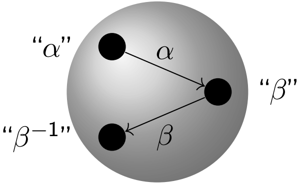
$$\langle \alpha, \beta \rangle \cong \mathbb{F}_2$$



# Examples

- Example 1: $\mathbb{Z} \curvearrowright S^1$, where $n \in \mathbb{Z}$ acts by rotation by $n$ radians. $\Pi_k = (k, \{0, 1\}, \mathcal{A})$, where $\mathcal{A} = \{c \mid c(0) \neq c(1)\}$. Proper $k$-colorings of $S^1$ are exactly $\Pi_k$-labelings of $S^1$.
- Example 2: $\mathbb{F}_2 \curvearrowright S^2$ (ignore the non-free part). Label each point with the rotation it will be moved by.

$$\langle \alpha, \beta \rangle \cong \mathbb{F}_2$$

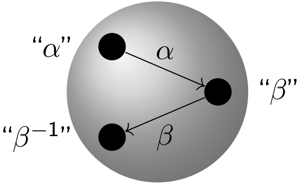



Each point needs to check that exactly two points will be moved to it.

# Examples

- Example 1: $\mathbb{Z} \curvearrowright S^1$, where $n \in \mathbb{Z}$ acts by rotation by $n$ radians. $\Pi_k = (k, \{0,1\}, \mathcal{A})$, where $\mathcal{A} = \{c \mid c(0) \neq c(1)\}$. Proper $k$-colorings of $S^1$ are exactly $\Pi_k$-labelings of $S^1$.
- Example 2: $\mathbb{F}_2 \curvearrowright S^2$ (ignore the non-free part). Label each point with the rotation it will be moved by.

$$\langle \alpha, \beta \rangle \cong \mathbb{F}_2$$




Each point needs to check that exactly two points will be moved to it.

- Example 3: Almost like example 2 but with $\mathbb{Z}^n \curvearrowright \mathbb{R}^2$ by translations.

# Complexity classes in descriptive combinatorics

Given an LCL $\Pi$ on $\Gamma$, we are interested in instances of the following sorts of questions:

- Let $\Gamma \curvearrowright X$ be a free Borel action on a standard Borel space. Does $X$ admit a Borel $\Pi$-labeling?

# Complexity classes in descriptive combinatorics

Given an LCL $\Pi$ on $\Gamma$, we are interested in instances of the following sorts of questions:

- Let $\Gamma \curvearrowright X$ be a free Borel action on a standard Borel space. Does $X$ admit a Borel $\Pi$-labeling?
- Let $\Gamma \curvearrowright (X, \mu)$ be a free Borel action on a stanard probability space. Does $X$ admit a $\mu$-measurable $\Pi$-labeling?
- Let $\Gamma \curvearrowright (X, \tau)$ be a free Borel action on a Polish space. Does $X$ admit a $\tau$-Baire measurable $\Pi$-labeling?

# Complexity classes in descriptive combinatorics

Given an LCL $\Pi$ on $\Gamma$, we are interested in instances of the following sorts of questions:

- Let $\Gamma \curvearrowright X$ be a free Borel action on a standard Borel space. Does $X$ admit a Borel $\Pi$-labeling?
- Let $\Gamma \curvearrowright (X, \mu)$ be a free Borel action on a stanard probability space. Does $X$ admit a $\mu$-measurable $\Pi$-labeling?
- Let $\Gamma \curvearrowright (X, \tau)$ be a free Borel action on a Polish space. Does $X$ admit a $\tau$-Baire measurable $\Pi$-labeling?
- Let $\Gamma \curvearrowright (X, \tau)$ be a free continuous action on a **zero-dimensional** Polish space. Does $X$ admit a continuous $\Pi$-labeling?

# Complexity classes in descriptive combinatorics

Given an LCL $\Pi$ on $\Gamma$, we are interested in instances of the following sorts of questions: These give rise to **complexity classes**.

- Let $\Gamma \curvearrowright X$ be a free Borel action on a standard Borel space. Does $X$ admit a Borel $\Pi$-labeling?
  If the answer is always yes, we say $\Pi \in \mathtt{BOREL}(\Gamma)$
- Let $\Gamma \curvearrowright (X, \mu)$ be a free Borel action on a stanard probability space. Does $X$ admit a $\mu$-measurable $\Pi$-labeling?
  If the answer is always yes, we say $\Pi \in \mathtt{MEASURE}(\Gamma)$
- Let $\Gamma \curvearrowright (X, \tau)$ be a free Borel action on a Polish space. Does $X$ admit a $\tau$-Baire measurable $\Pi$-labeling?
  If the answer is always yes, we say $\Pi \in \mathtt{BAIRE}(\Gamma)$
- Let $\Gamma \curvearrowright (X, \tau)$ be a free continuous action on a **zero-dimensional** Polish space. Does $X$ admit a continuous $\Pi$-labeling?
  If the answer is always yes, we say $\Pi \in \mathtt{CONTINUOUS}(\Gamma)$

# Complexity classes for $\mathbb{Z}$

We have the following picture, due to Grebík-Rozhoň ('23)

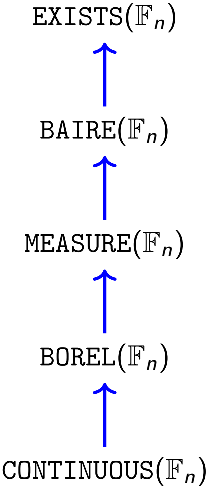
2-coloring $\in$ Exists a solution on $\mathbb{Z}$

$\uparrow \subsetneq$

3-coloring $\in$ $\texttt{CONTINUOUS}(\mathbb{Z}) = \texttt{BOREL}(\mathbb{Z}) = \texttt{MEASURE}(\mathbb{Z}) = \texttt{BAIRE}(\mathbb{Z}) = \dots$

# Complexity classes for $\mathbb{F}_n$ ($n \geq 2$)

- We have this picture, combining results of KST ('99), CMT-D ('16), Bernshteyn, BCGGRV ('22), CM ('16), and Marks('16).
- All inclusions are strict.



# Complexity classes for $\mathbb{F}_n$ ($n \geq 2$)

- We have this picture, combining results of KST ('99), CMT-D ('16), Bernshteyn, BCGGRV ('22), CM ('16), and Marks('16).
- All inclusions are strict.

| | |
|---|---|
| 2-coloring | $\texttt{EXISTS}(\mathbb{F}_n)$ |
| | $\uparrow$ |
| 3-coloring | $\texttt{BAIRE}(\mathbb{F}_n)$ |
| | $\uparrow$ |
| $2n$-coloring | $\texttt{MEASURE}(\mathbb{F}_n)$ |
| | $\uparrow$ |
| $2n$-coloring off lines | $\texttt{BOREL}(\mathbb{F}_n)$ |
| | $\uparrow$ |
| $2n+1$-coloring | $\texttt{CONTINUOUS}(\mathbb{F}_n)$ |

# Complexity classes for $\mathbb{F}_n$ ($n \geq 2$)

- We have this picture, combining results of KST ('99), CMT-D ('16), Bernshteyn, BCGGRV ('22), CM ('16), and Marks('16).
- All inclusions are strict.

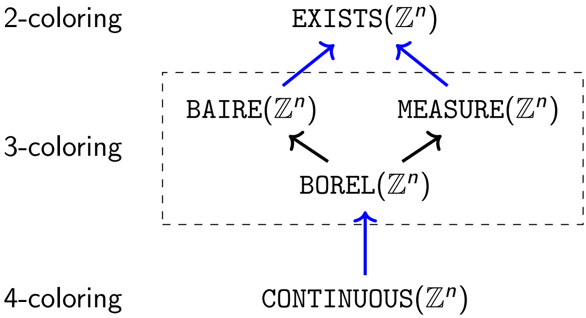
| | |
|---|---|
| 2-coloring | $\texttt{EXISTS}(\mathbb{F}_n)$ |
| | $\uparrow$ |
| 3-coloring, paradoxical decomp | $\texttt{BAIRE}(\mathbb{F}_n)$ |
| | $\uparrow$ |
| $2n$-coloring | $\texttt{MEASURE}(\mathbb{F}_n)$ |
| | $\uparrow$ |
| $2n$-coloring off lines | $\texttt{BOREL}(\mathbb{F}_n)$ |
| | $\uparrow$ |
| $2n+1$-coloring | $\texttt{CONTINUOUS}(\mathbb{F}_n)$ |

# Complexity classes for $\mathbb{F}_n$ ($n \geq 2$)

- We have this picture, combining results of KST ('99), CMT-D ('16), Bernshteyn, BCGGRV ('22), CM ('16), and Marks('16).
- All inclusions are strict.
- Note the nontrivial relationship $\mathtt{MEASURE}(\mathbb{F}_n) \subsetneq \mathtt{BAIRE}(\mathbb{F}_n)$

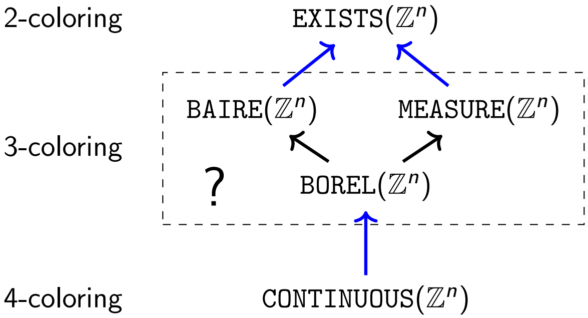
| | |
|---|---|
| 2-coloring | $\mathtt{EXISTS}(\mathbb{F}_n)$ |
| | $\uparrow$ |
| 3-coloring, paradoxical decomp | $\mathtt{BAIRE}(\mathbb{F}_n)$ |
| | $\uparrow$ |
| $2n$-coloring | $\mathtt{MEASURE}(\mathbb{F}_n)$ |
| | $\uparrow$ |
| $2n$-coloring off lines | $\mathtt{BOREL}(\mathbb{F}_n)$ |
| | $\uparrow$ |
| $2n+1$-coloring | $\mathtt{CONTINUOUS}(\mathbb{F}_n)$ |

# Complexity classes for $\mathbb{Z}^n$ ($n \geq 2$)




- The coloring results are due to Gao-Jackson-Krohne-Seward.

# Complexity classes for $\mathbb{Z}^n$ ($n \geq 2$)

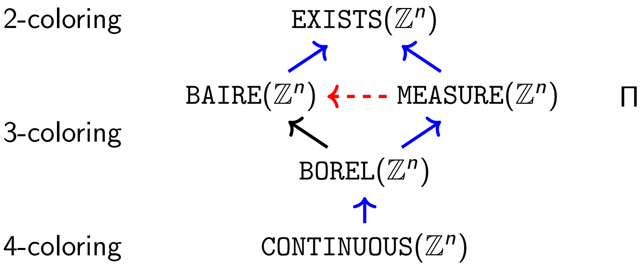



- The coloring results are due to Gao-Jackson-Krohne-Seward.

**Question (Grebík-Rozhoň '23)**

*Are any of the 3 complexity classes in this box distinct?*

# Main result

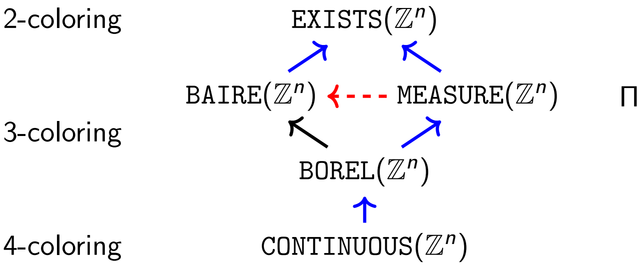



**Theorem (Berlow-Bernshteyn-Lyons-W.)**

*For every $n \geq 2$, there is an LCL $\Pi$ on $\mathbb{Z}^n$ such that:*

- $\Pi \in \mathtt{MEASURE}(\mathbb{Z}^n)$
- $\Pi \notin \mathtt{BAIRE}(\mathbb{Z}^n)$

# Main result




**Theorem (Berlow-Bernshteyn-Lyons-W.)**

*For every $n \geq 2$, there is an LCL $\Pi$ on $\mathbb{Z}^n$ such that:*

- $\Pi \in \texttt{MEASURE}(\mathbb{Z}^n)$
- $\Pi \notin \texttt{BAIRE}(\mathbb{Z}^n)$

This is the first time this non-inclusion was shown for **any** group.

# Outline

Descriptive Combinatorics

Complexity classes of LCLs

Connections with other areas

Toast

# Computable combinatorics

There are many areas of math in which people are interested in solving LCLs.

# Computable combinatorics

There are many areas of math in which people are interested in solving LCLs. One of the appealing features of this framework is that it lets us formalize connections between such settings. For example:

# Computable combinatorics

There are many areas of math in which people are interested in solving LCLs. One of the appealing features of this framework is that it lets us formalize connections between such settings. For example:

- *Computable combinatorics* is aesthetically quite similar to descriptive combinatorics: Given a computable graph on $\mathbb{N}$ and some combinatorial problem, we are interested in the computability of solutions to that problem.

# Computable combinatorics

There are many areas of math in which people are interested in solving LCLs. One of the appealing features of this framework is that it lets us formalize connections between such settings. For example:

- *Computable combinatorics* is aesthetically quite similar to descriptive combinatorics: Given a computable graph on $\mathbb{N}$ and some combinatorial problem, we are interested in the computability of solutions to that problem.
- By diagonalization, it is easy to build a computable, 2-regular, acyclic graph with no computable 2-coloring.

# Computable combinatorics

There are many areas of math in which people are interested in solving LCLs. One of the appealing features of this framework is that it lets us formalize connections between such settings. For example:

- *Computable combinatorics* is aesthetically quite similar to descriptive combinatorics: Given a computable graph on $\mathbb{N}$ and some combinatorial problem, we are interested in the computability of solutions to that problem.
- By diagonalization, it is easy to build a computable, 2-regular, acyclic graph with no computable 2-coloring.
- On the other hand, any such graph has a computable 3-coloring given by a greedy algorithm.

# Computable combinatorics

There are many areas of math in which people are interested in solving LCLs. One of the appealing features of this framework is that it lets us formalize connections between such settings. For example:

- Let's say $\Pi \in \texttt{COMPUTABLE}(\Gamma)$ if any free computable action $\Gamma \curvearrowright \mathbb{N}$ admits a computable $\Pi$-labeling.

# Computable combinatorics

There are many areas of math in which people are interested in solving LCLs. One of the appealing features of this framework is that it lets us formalize connections between such settings. For example:

- Let's say $\Pi \in \texttt{COMPUTABLE}(\Gamma)$ if any free computable action $\Gamma \curvearrowright \mathbb{N}$ admits a computable $\Pi$-labeling.

**Theorem (W. '24)**

*For any $n \in \mathbb{N}$, $\texttt{COMPUTABLE}(\mathbb{F}_n) = \texttt{BAIRE}(\mathbb{F}_n)$.*

# Factors of i.i.d.

- Probabilists are often interested in *invariant random $\Pi$-labelings* of a group $\Gamma$.

# Factors of i.i.d.

- Probabilists are often interested in *invariant random* $\Pi$-*labelings* of a group $\Gamma$.
- One simple invariant random process: Label each $g \in \Gamma$ with an independent random real $x(g) \in [0, 1]$.

# Factors of i.i.d.

- Probabilists are often interested in *invariant random $\Pi$-labelings* of a group $\Gamma$.
- One simple invariant random process: Label each $g \in \Gamma$ with an independent random real $x(g) \in [0, 1]$.
- We say $\Pi \in \texttt{FIID}(\Gamma)$ if a random $\Pi$-labeling can be obtained as a *factor* (pushforward) of this process. (*factor of i.i.d.*)

# Factors of i.i.d.

- Probabilists are often interested in *invariant random $\Pi$-labelings* of a group $\Gamma$.
- One simple invariant random process: Label each $g \in \Gamma$ with an independent random real $x(g) \in [0, 1]$.
- We say $\Pi \in \texttt{FIID}(\Gamma)$ if a random $\Pi$-labeling can be obtained as a *factor* (pushforward) of this process. (*factor of i.i.d.*)
- Equivalently, there is a measurable $\Pi$-labeling of the free part of the Bernoulli shift $c : [0, 1]^\Gamma \to \Lambda$.

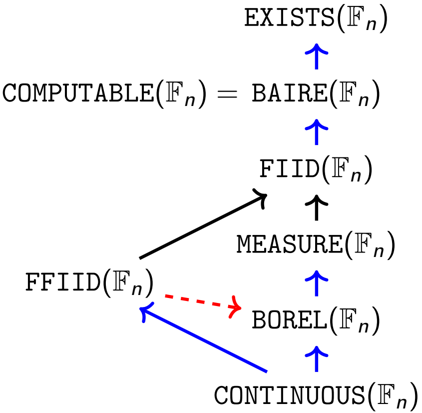
# Factors of i.i.d.

- Probabilists are often interested in *invariant random* $\Pi$*-labelings* of a group $\Gamma$.
- One simple invariant random process: Label each $g \in \Gamma$ with an independent random real $x(g) \in [0,1]$.
- We say $\Pi \in \mathtt{FIID}(\Gamma)$ if a random $\Pi$-labeling can be obtained as a *factor* (pushforward) of this process. (*factor of i.i.d.*)
- Equivalently, there is a measurable $\Pi$-labeling of the free part of the Bernoulli shift $c : [0,1]^\Gamma \to \Lambda$.
  Thus, $\mathtt{MEASURE}(\Gamma) \subseteq \mathtt{FIID}(\Gamma)$.

# Factors of i.i.d.

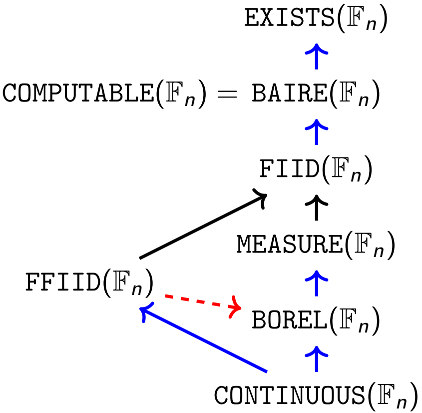
- Probabilists are often interested in *invariant random* $\Pi$*-labelings* of a group $\Gamma$.
- One simple invariant random process: Label each $g \in \Gamma$ with an independent random real $x(g) \in [0,1]$.
- We say $\Pi \in \mathtt{FIID}(\Gamma)$ if a random $\Pi$-labeling can be obtained as a *factor* (pushforward) of this process. (*factor of i.i.d.*)
- Equivalently, there is a measurable $\Pi$-labeling of the free part of the Bernoulli shift $c : [0,1]^\Gamma \to \Lambda$.
  Thus, $\mathtt{MEASURE}(\Gamma) \subseteq \mathtt{FIID}(\Gamma)$.
- We say $\Pi \in \mathtt{FFIID}(\Gamma)$ if for almost every $x \in [0,1]^\Gamma$, $c(x)$ depends on $x(g)$ for only finitely many $g \in \Gamma$. (*finitary* factor)

# More complexity classes for $\mathbb{F}_n$ ($n \geq 2$)



# More complexity classes for $\mathbb{F}_n$ ($n \geq 2$)

- The class $\mathtt{FFIID}(\mathbb{F}_n)$ is poorly understood.

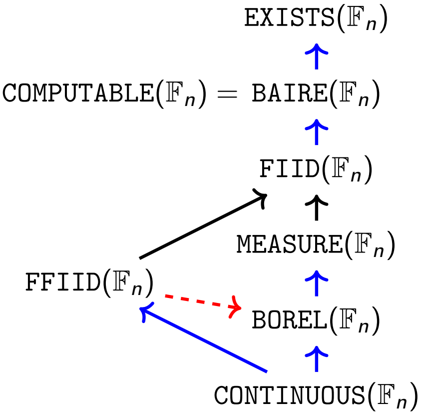


# More complexity classes for $\mathbb{F}_n$ ($n \geq 2$)

- The class $\texttt{FFIID}(\mathbb{F}_n)$ is poorly understood.

**Question**

*Does* $\texttt{FFIID}(\Gamma) = \texttt{FIID}(\Gamma)$ *for every countable group* $\Gamma$*?*



# More complexity classes for $\mathbb{F}_n$ ($n \geq 2$)

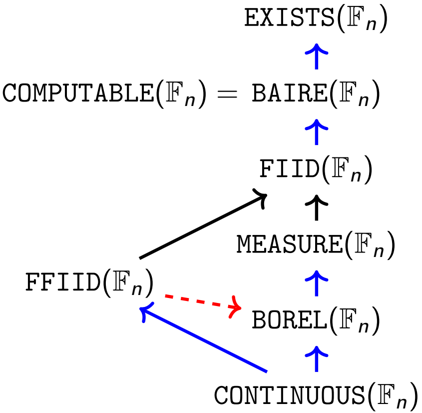
- The class $\texttt{FFIID}(\mathbb{F}_n)$ is poorly understood.

**Question**

*Does* $\texttt{FFIID}(\Gamma) = \texttt{FIID}(\Gamma)$ *for every countable group* $\Gamma$*?*

**Question**

*Does* $\texttt{COMPUTABLE}(\Gamma) = \texttt{BAIRE}(\Gamma)$ *for every "computable" group* $\Gamma$*?*



# Main result extended

**Theorem (Berlow-Bernshteyn-Lyons-W.)**

*For every $n \geq 2$, there is an LCL $\Pi$ on $\mathbb{Z}^n$ such that:*

- $\Pi \in \mathtt{MEASURE}(\mathbb{Z}^n)$
- $\Pi \in \mathtt{COMPUTABLE}(\mathbb{Z}^n)$
- $\Pi \notin \mathtt{BAIRE}(\mathbb{Z}^n)$
- $\Pi \notin \mathtt{FFIID}(\mathbb{Z}^n)$

# Main result extended

**Theorem (Berlow-Bernshteyn-Lyons-W.)**

*For every $n \geq 2$, there is an LCL $\Pi$ on $\mathbb{Z}^n$ such that:*

- $\Pi \in \mathtt{MEASURE}(\mathbb{Z}^n)$
- $\Pi \in \mathtt{COMPUTABLE}(\mathbb{Z}^n)$
- $\Pi \notin \mathtt{BAIRE}(\mathbb{Z}^n)$
- $\Pi \notin \mathtt{FFIID}(\mathbb{Z}^n)$

- Modifications of $\Pi$ allow us to obtain even more separations...

# Outline

Descriptive Combinatorics

Complexity classes of LCLs

Connections with other areas

Toast

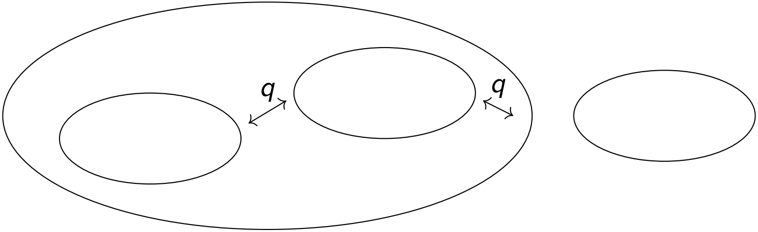
# Toast

Fix a free action $\mathbb{Z}^n \curvearrowright X$. Let dist denote the path metric of the associated graph on $X$.

**Definition (Gao-Jackson-Krohne-Seward)**

Let $q \in \mathbb{N}^+$. A $q$-**toast** on $X$ is a collection $\mathcal{T} \subseteq [X]^{<\infty}$ such that for all $C \neq D \in \mathcal{T}$, either

- $\text{dist}(C, D) > q$
- $\text{dist}(C, X \setminus D) > q$
- $\text{dist}(D, X \setminus C) > q$.



# Toast and coloring

We say a toast $\mathcal{T}$ is **complete** if $\bigcup \mathcal{T} = X$.

# Toast and coloring

We say a toast $\mathcal{T}$ is **complete** if $\bigcup \mathcal{T} = X$.

**Theorem (Gao-Jackson-Krohne-Seward)**

*Let $\mathbb{Z}^n \curvearrowright X$ be a free Borel action. For any $q$, $X$ admits a Borel complete $q$-toast.*

# Toast and coloring

We say a toast $\mathcal{T}$ is **complete** if $\bigcup \mathcal{T} = X$.

**Theorem (Gao-Jackson-Krohne-Seward)**

*Let $\mathbb{Z}^n \curvearrowright X$ be a free Borel action. For any $q$, $X$ admits a Borel complete $q$-toast.*

**Corollary (Gao-Jackson-Krohne-Seward)**

*3-coloring* $\in \mathtt{BOREL}(\mathbb{Z}^n)$.

# Toast and coloring

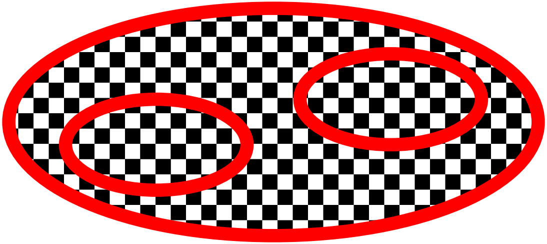
We say a toast $\mathcal{T}$ is **complete** if $\bigcup \mathcal{T} = X$.

**Theorem (Gao-Jackson-Krohne-Seward)**

*Let $\mathbb{Z}^n \curvearrowright X$ be a free Borel action. For any $q$, $X$ admits a Borel complete $q$-toast.*

**Corollary (Gao-Jackson-Krohne-Seward)**

*3-coloring* $\in \mathtt{BOREL}(\mathbb{Z}^n)$.



# Rectangular toast

- All positive results we knew about LCLs on $\mathbb{Z}^n$ used similar toast-based constructions.

# Rectangular toast

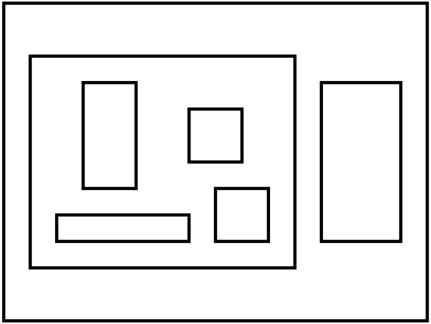
- All positive results we knew about LCLs on $\mathbb{Z}^n$ used similar toast-based constructions.
- Instead of changing this, we consider what can be achieved by using slightly nicer toasts.

# Rectangular toast

- All positive results we knew about LCLs on $\mathbb{Z}^n$ used similar toast-based constructions.
- Instead of changing this, we consider what can be achieved by using slightly nicer toasts.

**Definition**

A **rectangular toast** is a toast whose pieces are all rectangles.



# Existence of rectangular toast

**Proposition (Gao-Jackson-Krohne-Seward)**

*There exists a free Borel action $\mathbb{Z}^n \curvearrowright X$ Which does not admit a Borel complete rectangular 1-toast.*

# Existence of rectangular toast

**Proposition (Gao-Jackson-Krohne-Seward)**

*There exists a free Borel action $\mathbb{Z}^n \curvearrowright X$ Which does not admit a Borel complete rectangular 1-toast.*

- This is a category argument.

# Existence of rectangular toast

**Proposition (Gao-Jackson-Krohne-Seward)**

*There exists a free Borel action $\mathbb{Z}^n \curvearrowright X$ Which does not admit a Borel complete rectangular 1-toast.*

- This is a category argument. For $\mathbb{Z}^n \curvearrowright S^1$ by $n$ independent irrational rotations and $\mathcal{T}$ a Borel rectangular 1-toast, one can show $\bigcup \mathcal{T}$ is not comeager.

# Existence of rectangular toast

**Proposition (Gao-Jackson-Krohne-Seward)**

*There exists a free Borel action $\mathbb{Z}^n \curvearrowright X$ Which does not admit a Borel complete rectangular 1-toast.*

- This is a category argument. For $\mathbb{Z}^n \curvearrowright S^1$ by $n$ independent irrational rotations and $\mathcal{T}$ a Borel rectangular 1-toast, one can show $\bigcup \mathcal{T}$ is not comeager.

**Proposition (Folklore?)**

*Let $\mathbb{Z}^n \curvearrowright X$ be a free Borel action and $\mu \in P(X)$. For any $q$, $X$ admits a Borel rectangular $q$-toast $\mathcal{T}$ with $\mu(\bigcup \mathcal{T}) = 1$.*

# Rectangular toast in other areas

**Proposition (Folklore?)**

*Let $\mathbb{Z}^n \curvearrowright X$ be a free Borel action and $\mu \in P(X)$. For any $q$, $X$ admits a Borel rectangular $q$-toast $\mathcal{T}$ with $\mu(\bigcup \mathcal{T}) = 1$.*

# Rectangular toast in other areas

**Proposition (Folklore?)**

*Let $\mathbb{Z}^n \curvearrowright X$ be a free Borel action and $\mu \in P(X)$. For any $q$, $X$ admits a Borel rectangular $q$-toast $\mathcal{T}$ with $\mu(\bigcup \mathcal{T}) = 1$.*

**Proposition (Berlow-Bernshteyn-Lyons-W.)**

*An invariant random rectangular 1-toast on $\mathbb{Z}^n$ cannot be a* **finitary** *factor of i.i.d.*

# Rectangular toast in other areas

## Proposition (Folklore?)

*Let $\mathbb{Z}^n \curvearrowright X$ be a free Borel action and $\mu \in P(X)$. For any $q$, $X$ admits a Borel rectangular $q$-toast $\mathcal{T}$ with $\mu(\bigcup \mathcal{T}) = 1$.*

## Proposition (Berlow-Bernshteyn-Lyons-W.)

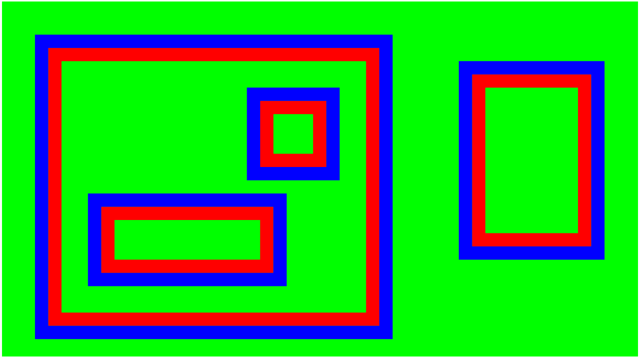
*An invariant random rectangular 1-toast on $\mathbb{Z}^n$ cannot be a* **finitary** *factor of i.i.d.*

## Proposition

*Any free computable action $\mathbb{Z}^n \curvearrowright \mathbb{N}$ admits a computable rectangular $q$-toast for any $q$.*

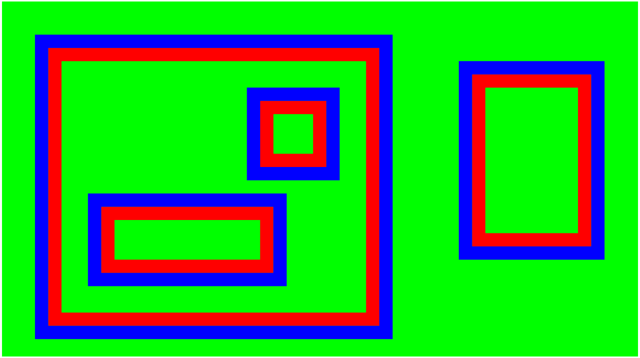
# The rectangular toast LCL

Given a rectangular $10n$-toast $\mathcal{T}$ on $X$, let $f_{\mathcal{T}} : X \to \{R, B, G\}$ be defined as in the picture.



# The rectangular toast LCL

Given a rectangular $10n$-toast $\mathcal{T}$ on $X$, let $f_{\mathcal{T}} : X \to \{R, B, G\}$ be defined as in the picture.



## Definition

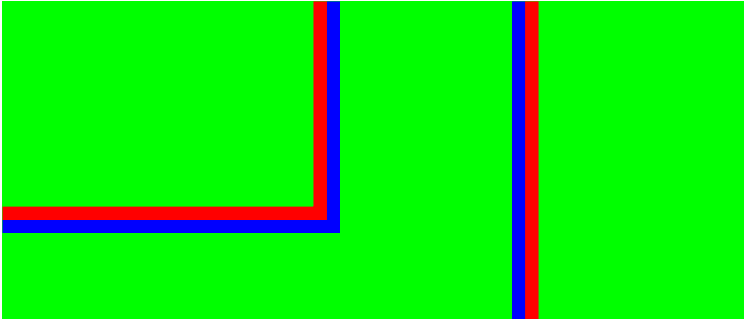
Let $\Pi_{RT}$ be the LCL on $\mathbb{Z}^n$ with window $[0, 10n]^n$ and labels $\{R, B, G\}$ whose valid configurations are those that look locally like $f_{\mathcal{T}}$ for some rectangular $10n$-toast $\mathcal{T}$.

# The key lemma

If $f : X \to \{R, B, G\}$ is a $\Pi_{RT}$-labeling, we need not have $f = f_{\mathcal{T}}$ for some rectangular toast $\mathcal{T}$.

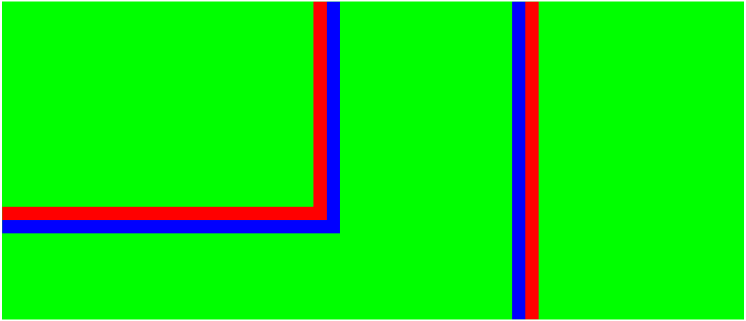
# The key lemma

If $f : X \to \{R, B, G\}$ is a $\Pi_{RT}$-labeling, we need not have $f = f_{\mathcal{T}}$ for some rectangular toast $\mathcal{T}$.



# The key lemma

If $f : X \to \{R, B, G\}$ is a $\Pi_{RT}$-labeling, we need not have $f = f_{\mathcal{T}}$ for some rectangular toast $\mathcal{T}$.

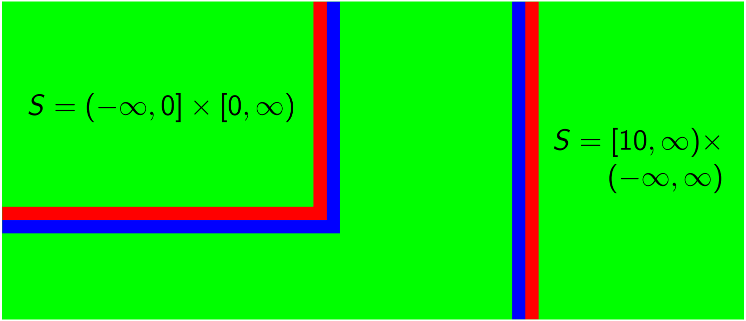


**Lemma**

*Let $f$ be a $\Pi_{RT}$-labeling. Let $C$ be a connected component of $f^{-1}(R)$. Then $C = \partial S$ for some product of (possibly infinite) intervals $S$.*

# The key lemma

If $f : X \to \{R, B, G\}$ is a $\Pi_{RT}$-labeling, we need not have $f = f_{\mathcal{T}}$ for some rectangular toast $\mathcal{T}$.




**Lemma**

*Let $f$ be a $\Pi_{RT}$-labeling. Let $C$ be a connected component of $f^{-1}(R)$. Then $C = \partial S$ for some product of (possibly infinite) intervals $S$.*

# The colored rectangular toast LCL

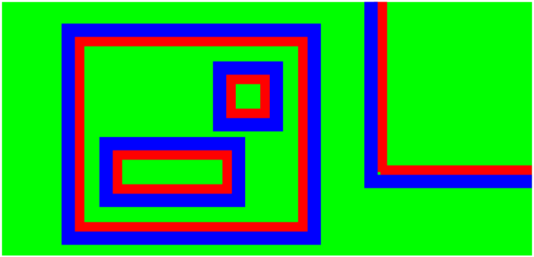
- Note that the constant $G$ function is a solution to $\Pi_{RT}$.

# The colored rectangular toast LCL

- Note that the constant $G$ function is a solution to $\Pi_{RT}$.

**Definition**

Let $\Pi_{CRT}$ be the LCL on $\mathbb{Z}^n$ with labels $\{R, B, 0, 1\}$ encoding the following problem: "Solve $\Pi_{RT}$, then 2-color the green points with $\{0, 1\}$".

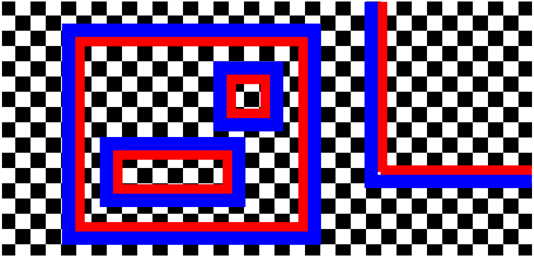


# The colored rectangular toast LCL

- Note that the constant $G$ function is a solution to $\Pi_{RT}$.

**Definition**

Let $\Pi_{CRT}$ be the LCL on $\mathbb{Z}^n$ with labels $\{R, B, 0, 1\}$ encoding the following problem: "Solve $\Pi_{RT}$, then 2-color the green points with $\{0, 1\}$".



# Proof of the main result

**Theorem (Berlow-Bernshteyn-Lyons-W.)**

- $\Pi_{CRT} \in \mathtt{MEASURE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \in \mathtt{COMPUTABLE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \notin \mathtt{BAIRE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \notin \mathtt{FFIID}(\mathbb{Z}^n)$

# Proof of the main result

**Theorem (Berlow-Bernshteyn-Lyons-W.)**

- $\Pi_{CRT} \in \mathtt{MEASURE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \in \mathtt{COMPUTABLE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \notin \mathtt{BAIRE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \notin \mathtt{FFIID}(\mathbb{Z}^n)$

- It is easy to use a definable rectangular toast to produce a similarly definable $\Pi_{CRT}$-labeling.

# Proof of the main result

**Theorem (Berlow-Bernshteyn-Lyons-W.)**

- $\Pi_{CRT} \in \mathtt{MEASURE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \in \mathtt{COMPUTABLE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \notin \mathtt{BAIRE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \notin \mathtt{FFIID}(\mathbb{Z}^n)$

# Proof of the main result

**Theorem (Berlow-Bernshteyn-Lyons-W.)**

- $\Pi_{CRT} \in \texttt{MEASURE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \in \texttt{COMPUTABLE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \notin \texttt{BAIRE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \notin \texttt{FFIID}(\mathbb{Z}^n)$

- To show $\Pi_{CRT} \notin \texttt{BAIRE}(\mathbb{Z}^n)$, let $\mathbb{Z}^n \curvearrowright S^1$ by independent irrational rotations, and suppose $S^1$ has a Baire measurable $\Pi_{CRT}$-labeling.

# Proof of the main result

**Theorem (Berlow-Bernshteyn-Lyons-W.)**

- $\Pi_{CRT} \in \mathtt{MEASURE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \in \mathtt{COMPUTABLE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \notin \mathtt{BAIRE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \notin \mathtt{FFIID}(\mathbb{Z}^n)$

- To show $\Pi_{CRT} \notin \mathtt{BAIRE}(\mathbb{Z}^n)$, let $\mathbb{Z}^n \curvearrowright S^1$ by independent irrational rotations, and suppose $S^1$ has a Baire measurable $\Pi_{CRT}$-labeling.
- With arguments similar to those from the beginning of the talk, one can show that on the generic orbit, any connected component of 2-colored points intersects any axis-parallel line only finitely often.

# Proof of the main result

**Theorem (Berlow-Bernshteyn-Lyons-W.)**

- $\Pi_{CRT} \in \texttt{MEASURE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \in \texttt{COMPUTABLE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \notin \texttt{BAIRE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \notin \texttt{FFIID}(\mathbb{Z}^n)$

- To show $\Pi_{CRT} \notin \texttt{BAIRE}(\mathbb{Z}^n)$, let $\mathbb{Z}^n \curvearrowright S^1$ by independent irrational rotations, and suppose $S^1$ has a Baire measurable $\Pi_{CRT}$-labeling.
- With arguments similar to those from the beginning of the talk, one can show that on the generic orbit, any connected component of 2-colored points intersects any axis-parallel line only finitely often.
- First, this rules out "infinitary rectangles".

# Proof of the main result

**Theorem (Berlow-Bernshteyn-Lyons-W.)**

- $\Pi_{CRT} \in \texttt{MEASURE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \in \texttt{COMPUTABLE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \notin \texttt{BAIRE}(\mathbb{Z}^n)$
- $\Pi_{CRT} \notin \texttt{FFIID}(\mathbb{Z}^n)$

- To show $\Pi_{CRT} \notin \texttt{BAIRE}(\mathbb{Z}^n)$, let $\mathbb{Z}^n \curvearrowright S^1$ by independent irrational rotations, and suppose $S^1$ has a Baire measurable $\Pi_{CRT}$-labeling.
- With arguments similar to those from the beginning of the talk, one can show that on the generic orbit, any connected component of 2-colored points intersects any axis-parallel line only finitely often.
- First, this rules out "infinitary rectangles".
- Second, it forces the encoded toast to be complete.

# Thanks for listening!