Strong indivisibility for graphs

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Ramsey theory has been a boon to our field

Definition. A relational structure M is *indivisible* if for every $c : M \to k$, there is an infinite monochromatic $H \subseteq M$ such that $H \cong M$.

- (RT^1). $\mathbb{N}$ is indivisible in the empty language: for every $c : \mathbb{N} \to k$, there is an infinite monochromatic H.

  $\mathrm{RCA}_0 \vdash \mathrm{RT}^1 \leftrightarrow B\Sigma^0_2$ (Hirst)

- (TT^1). As a poset, $(2^{<\mathbb{N}}, \subseteq)$ is indivisible.

  $\mathrm{RCA}_0 \vdash I\Sigma^0_2 \rightarrow \mathrm{TT}^1 \rightarrow B\Sigma^0_2$ (Chubb, Hirst, McNicholl)

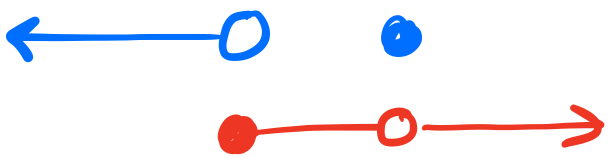
  (Chong, Li, Wang, Yang) (Corduan, Groszek, Mileti)

- The linear order $(\mathbb{Q}, \leq)$ is indivisible.
- The random graph $\mathcal{R}$ is indivisible.

Fact. For RT^1 and $\mathcal{R}$, one of the full colors is always homogeneous.

This is not true for $(\mathbb{Q}, \leq)$ and TT^1.

For $(\mathbb{Q}, \leq)$

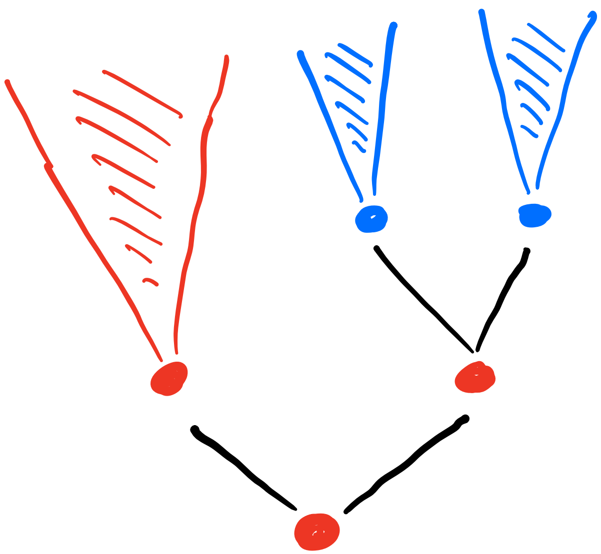


BLUE has greatest element

RED has least element

$\therefore$ neither $\cong \mathbb{Q}$

For $(2^{<\omega}, \subseteq)$



Strong indivisibility

Definition. M is *strongly indivisible* if for all partitions $M = X_0 \sqcup X_1$, $X_i \cong M$ for some $i < 2$.

The (countable) strongly indivisible structures have been classified for

- graphs (Cameron),
- linear and partial orders, tournaments (Bonato, Cameron and Delić),
- certain types of Fraïssé limits (Bonato and Delić).

Theorem (Cameron). A countable graph G is strongly indivisible if and only if G is isomorphic to K_ω, $\overline{K}_\omega$ or $\mathcal{R}$.

Theorem. (Bonato, Cameron and Delić) A partial order P is strongly indivisible if and only if P is an infinite antichain or P is isomorphic to ω^α or $(\omega^\alpha)^*$ for some ordinal $\alpha > 0$.

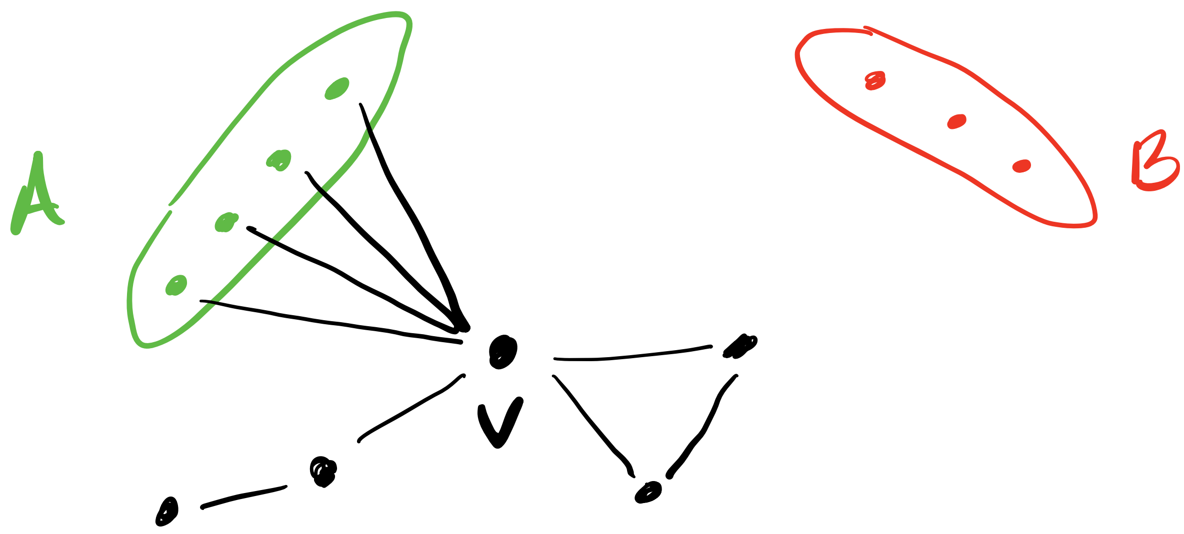
Random graph

Let G be a (countable) graph and $A, B \subseteq G$ be finite.

$$\langle A, B \rangle \text{ is an } n\text{-pair} \iff |A| + |B| = n \text{ and } A \cap B = \emptyset$$

For each n-pair $\langle A, B \rangle$, $\langle A, B \rangle$ is extendible if and only if

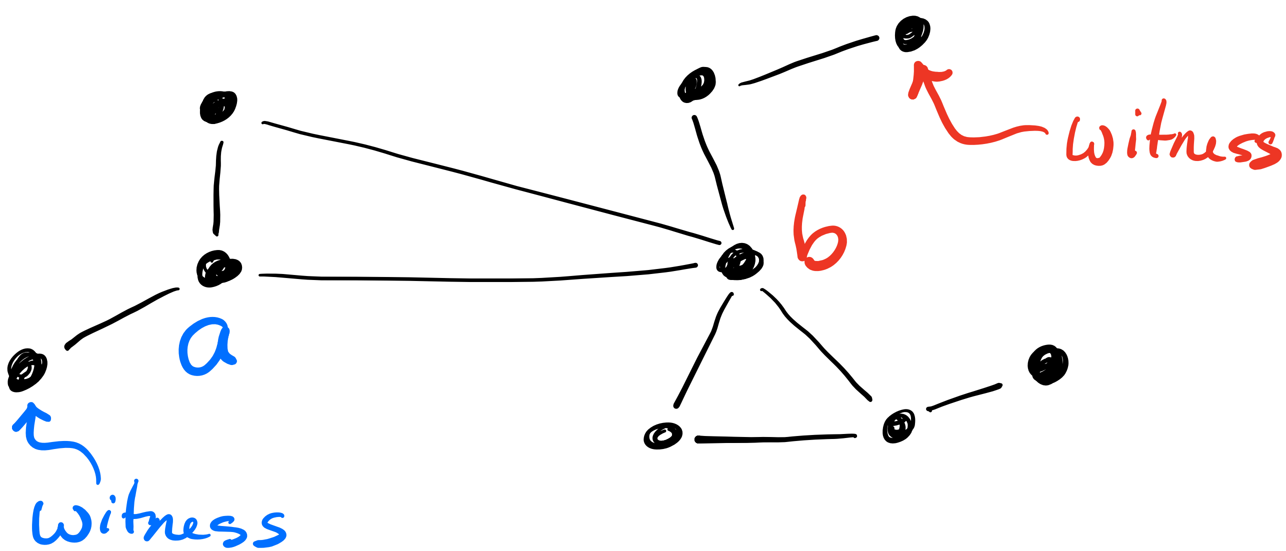
$$(\exists v \in G - (A \cup B))\left((\forall a \in A)E(a, v) \wedge (\forall b \in B)\neg E(b, v)\right)$$



Example. There is one 0-pair, $\langle \emptyset, \emptyset \rangle$, and it is extendible.

Example. There are two types of 1-pairs: $\langle \{a\}, \emptyset \rangle$ and $\langle \emptyset, \{b\} \rangle$.

- $\langle \{a\}, \emptyset \rangle$ is extendible unless a is isolated (not connected to anything).
- $\langle \emptyset, \{b\} \rangle$ is extendible unless b is universal (connected to everything).

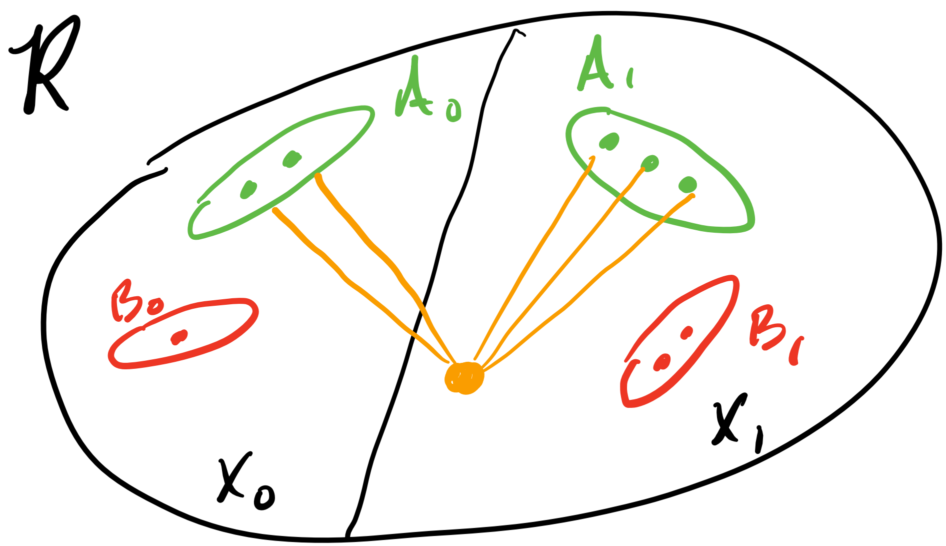


G is a *random graph* if and only if every n-pair in G is extendible.

Fact. By a back-and-forth argument, if G and H are (countable) random graphs, then $G \cong H$. In fact, the random graph $\mathcal{R}$ is computably categorical and is unambiguous in RCA_0.

One direction of Cameron's theorem holds in RCA_0: K_ω, $\overline{K_\omega}$ and $\mathcal{R}$ are strongly indivisible.

Why does it hold for $\mathcal{R}$? Suppose $\mathcal{R} = X_0 \sqcup X_1$ with $X_0, X_1 \not\cong \mathcal{R}$. For $i < 2$, fix $\langle A_i, B_i \rangle$ which is not extendible in X_i.



Look at witness point for $\langle A_0 \cup A_1, B_0 \cup B_1 \rangle$ is extendible in $\mathcal{R}$.

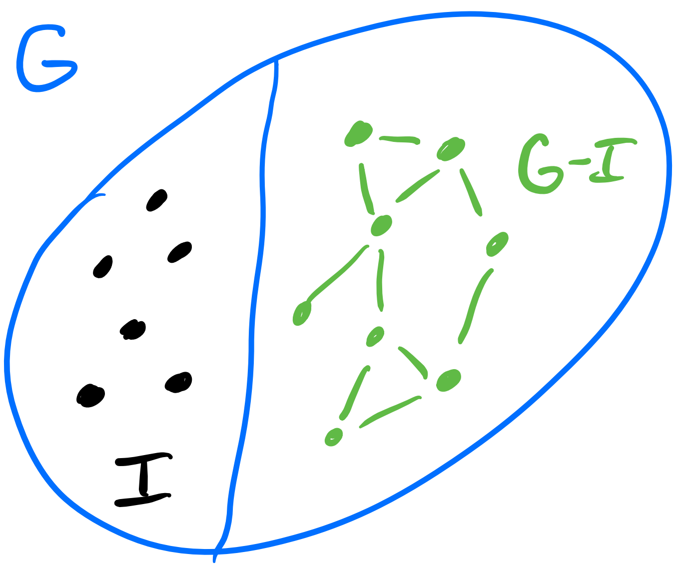
Cameron's classical proof for the other direction.

Suppose $G \ncong K_\omega, \overline{K}_\omega, \mathcal{R}$ is countable and not finite.

Goal: Describe a partition $G = X_0 \sqcup X_1$ such that $X_0, X_1 \ncong G$.

Case 1. G has isolated nodes. Let $I =$ set of isolated nodes.

$$G = I \sqcup (G \setminus I)$$



- $I \cong \overline{K}_n$ or $\overline{K}_\omega$, so $I \ncong G$ because G infinite and $G \ncong \overline{K}_\omega$
- $G - I$ has no isolated points, so $G \ncong G - I$

Case 2. G has universal nodes. Let $U =$ set of universal nodes. By the same argument,

$$G = U \sqcup (G \setminus U) \text{ with } G \ncong U, (G \setminus U)$$

using the fact that $G \ncong K_\omega$.

Case 3. G has neither isolated nor universal nodes.

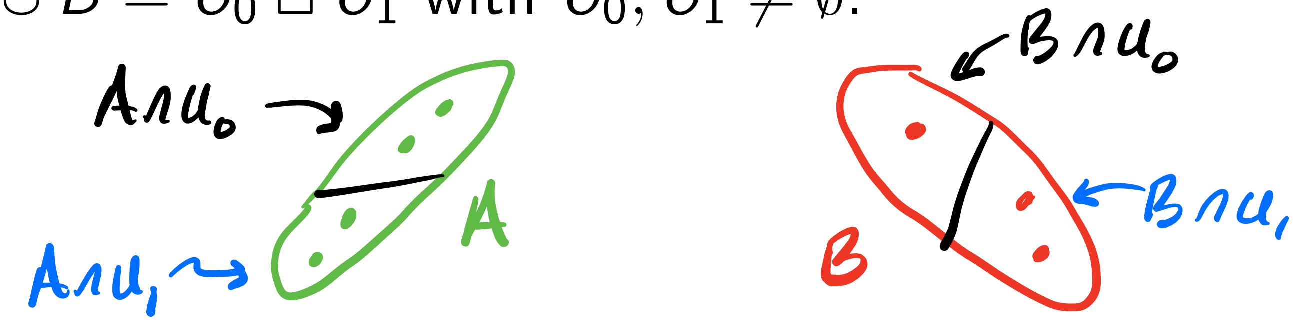
Since $G \ncong \mathcal{R}$, fix least n s.t. there is a non-extendible n-pair $\langle A, B \rangle$.

- $n \neq 0$ because the only 0-pair is $\langle \emptyset, \emptyset \rangle$, which is extendible.
- $n \neq 1$ because $\langle \{a\}, \emptyset \rangle$ and $\langle \emptyset, \{b\} \rangle$ are extendible.

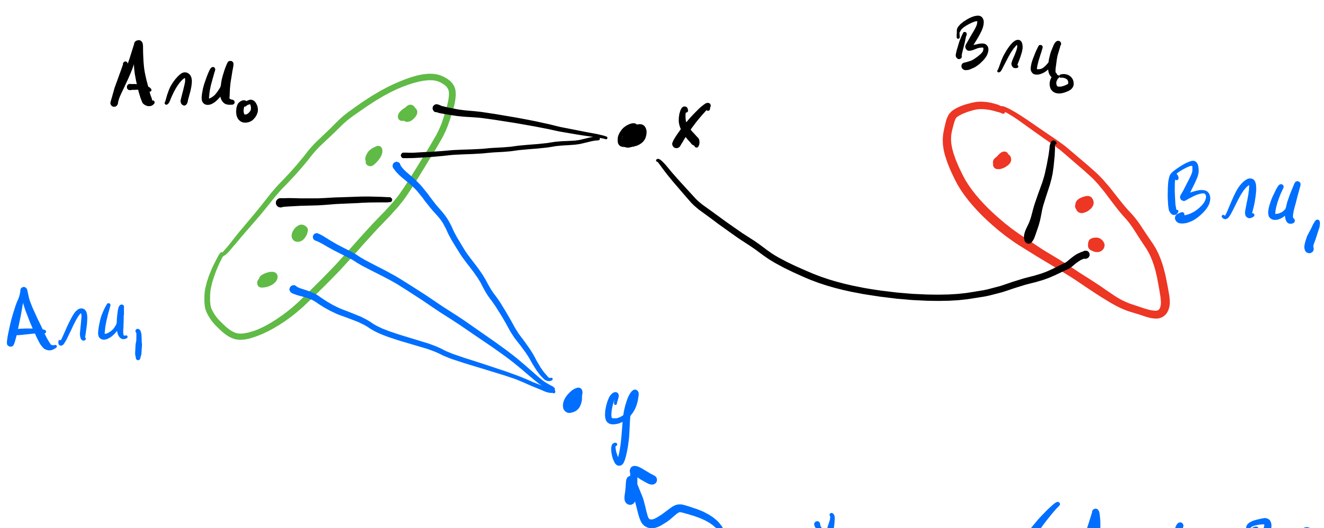
not universal

not isolated

Fix n-pair $\langle A, B\rangle$ that is not extendible in G. Since $n \geq 2$, we can write $A \cup B = U_0 \sqcup U_1$ with $U_0, U_1 \neq \emptyset$.



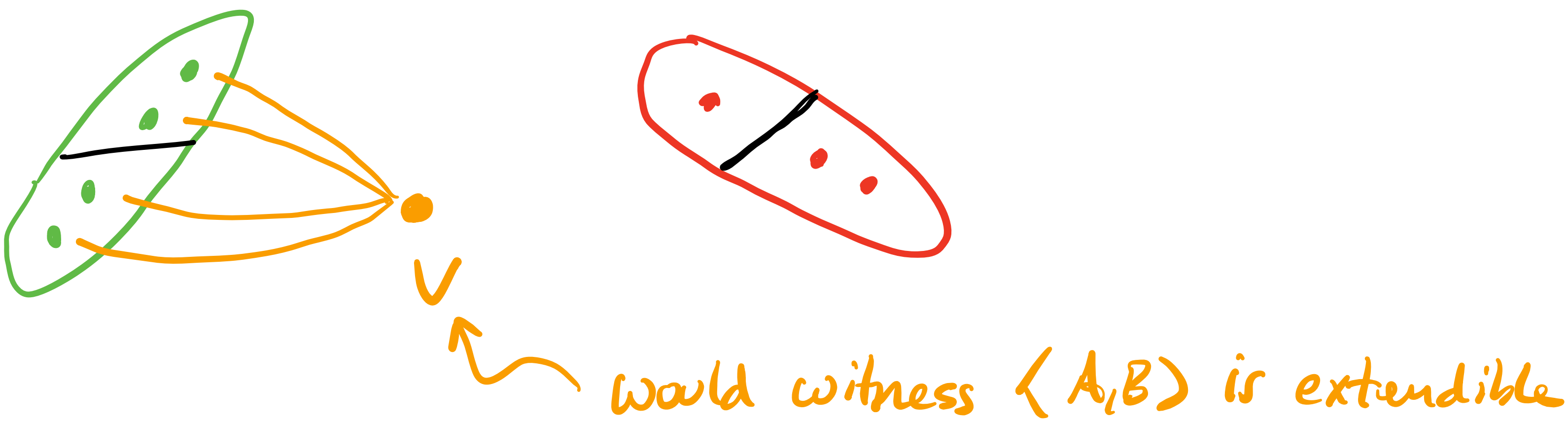
The pairs $\langle A \cap U_0, B \cap U_0\rangle$ and $\langle A \cap U_1, B \cap U_1\rangle$ are extendible in G by the minimality of n.



witnesses $\langle A \cap U_1, B \cap U_1\rangle$ is extendible

A vertex v is *correctly joined to* U_i if v witnesses the extendibility of $\langle A \cap U_i, B \cap U_i \rangle$ in G.

Key observation. No vertex v can be correctly joined to U_0 and U_1 since $\langle A, B \rangle$ is not extendible.



Goal: to define a partition $G = X_0 \sqcup X_1$ such that $G \ncong X_i$.

$$X_0 = U_0 \cup \{v : v \notin U_1 \text{ and is not correctly joined to } U_0\}$$
$$X_1 = (G \setminus X_0)$$

By the key observation,

$$X_1 \subseteq U_1 \cup \{v : v \text{ is not correctly joined to } U_1\}$$

Therefore, each $\langle A \cap U_i, B \cap U_i \rangle$ is not extendible in X_i, so $X_i \ncong G$ by minimality of n.

Note. Suppose G is computable in this case. After non-uniformly fixing the non-extendible n-pair $\langle A, B \rangle$, the partition $G = X_0 \sqcup X_1$ is computable.

What was used in this proof?

Case 1. $G = I \sqcup (G \setminus I)$ where I is the set of isolated vertices.

Exercise. Over RCA_0, ACA_0 is equivalent to the existence of the set of isolated nodes for every graph G.

Case 2. $G = U \sqcup (G \setminus U)$ where U is the set of universal vertices.

Case 3. Use the least n s.t. $\underbrace{\text{there is a non-extendible } n\text{-pair } \langle A, B \rangle}_{\Sigma^0_2}$.

Theorem. (Dzhafarov, Solomon, Volpi) Over RCA_0, TFAE

- $\mathsf{L}\Sigma^0_2$ (which is equivalent to $\mathsf{I}\Sigma^0_2$), and
- For every $G \ncong \mathcal{R}$, there is least n s.t. G has a non-extendible n-pair.

Question. Is there a proof that works in RCA_0? What about in REC?

Working in REC

Does REC satisfy that for every $G \not\cong K_\omega, \overline{K}_\omega, \mathcal{R}$, there is a partition $G = X_0 \sqcup X_1$ such that $X_i \not\cong G$?

(Can remove)

To address this question, fix a computable graph G that is not computably isomorphic to $K_\omega, \overline{K}_\omega, \mathcal{R}$.

Question: Is there a *computable* partition $G = X_0 \sqcup X_1$ such that neither X_0 nor X_1 is *computably* isomorphic to G?

Note. If G has no isolated or universal vertices, then we can non-uniformly fix a non-extendible n-pair $\langle A, B \rangle$ and define a computable partition $G = X_0 \sqcup X_1$ such that neither X_0 nor X_1 is *classically* isomorphic to G.

Hope: Is there a *computable* partition $G = X_0 \sqcup X_1$ such that neither X_0 nor X_1 is *classically* isomorphic to G?

Answer 1. Yes, as long as we view G only up to computable presentation.

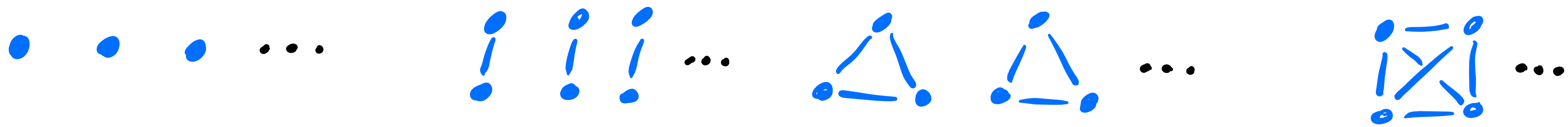
Theorem. (Dzhafarov, Solomon, Volpi) For every computable graph $G \not\cong K_\omega, \overline{K}_\omega, \mathcal{R}$, there is a computable $H \cong G$ and a computable partition $H = X_0 \sqcup X_1$ s.t. neither X_0 nor X_1 is *classically* isomorphic to G.

Why? Just run Cameron's argument using the following theorem.

Theorem. (Dzhafarov, Solomon, Volpi) Every computable graph has a computable copy in which the set of isolated (or universal) nodes is computable.

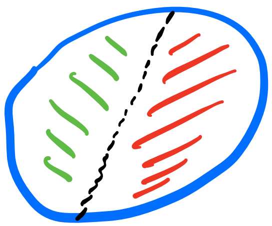
Answer 2. No, if we cannot shift the presentation of G.

Let $K^{\infty}_{<\omega}$ be the graph consisting of infinitely many K_n for each n.



Theorem. (Dzhafarov, Solomon, Volpi) There is computable $G \cong K^{\infty}_{<\omega}$ s.t. for every computable partition $G = X_0 \sqcup X_1$, either X_0 or X_1 is classically isomorphic to $K^{\infty}_{<\omega}$.

Split K_n:



= K_m

= K_p with $p + m = n$

If X_i contains infinitely many K_n for each n, then $X_i \cong G$ regardless of what else it contains.

Second Hope. Can we strengthen this result to require either X_0 or X_1 is computably isomorphic to G?

Back to the main question about REC

Fix a computable $G \not\cong K_\omega, \overline{K}_\omega, \mathcal{R}$. Is there a computable partition $G = X_0 \sqcup X_1$ such that neither X_0 nor X_1 is computably isomorphic to G?

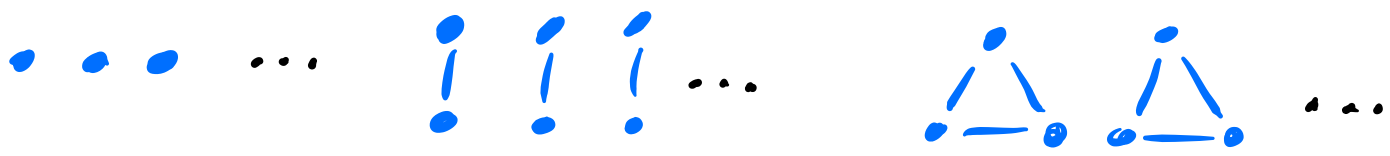
Two observations:

- If G has no isolated or universal vertices, then we define $G = X_0 \sqcup X_1$ computably using an appropriate non-extendible n-pair.
- If G has universal vertices, then we can move from G to $\overline{G}$ by swapping edges and non-edges. This turns universal vertices into isolated vertices, but doesn't change which partitions work.

Therefore, we can reduce to the case when $G \not\cong \overline{K}_\omega$ is a computable graph with isolated vertices.

Theorem. (Dzhafarov, Solomon, Volpi) Let $G \not\cong \overline{K}_\omega$ be a computable graph with isolated vertices. If the set of vertices with finite degree is c.e. then there is a computable partition $G = X_0 \sqcup X_1$ s.t. neither X_0 nor X_1 is computably isomorphic to G.

Example. Consider the earlier graph $K^{\infty}_{<\omega}$.



The theorem applies to every computable copy of $K^{\infty}_{<\omega}$ since all nodes in this graph have finite degree.

We constructed a computable copy G such that for every computable partition $G = X_0 \sqcup X_1$, either X_0 or X_1 is classically isomorphic to G.

However, by this theorem, there are computable partitions $G = X_0 \sqcup X_1$ such that neither X_0 nor X_1 is computably isomorphic to G.

Theorem. (Dzhafarov, Solomon, Volpi) Let $G \not\cong \overline{K}_\omega$ be a computable graph with isolated vertices. If the set of vertices with finite degree is c.e. then there is a computable partition $G = X_0 \sqcup X_1$ s.t. neither X_0 nor X_1 is computably isomorphic to G.

Proof sketch. If I is computable, set $G = I \sqcup (G \setminus I)$, so assume not.

At stage s, we must put s into X_0 or X_1. We need to meet

$$R_e : \Phi_e \text{ is not isomorphism from } G \text{ onto } X_0 \text{ or } X_1.$$

Phase 1. Do nothing until $\Phi_e(0)$ converges, then set i s.t. $\Phi_e(0) \in X_i$.

To stop $\Phi_i : G \to X_i$ being an isomorphism, we want to make Φ_e map an isolated point in G to a non-isolated point in X_i (or vice versa).

To assist our strategy, we threaten to compute the set of isolated nodes.

Phase 2. Set x_e to be the least node in G_s that looks isolated.

- Declare that all $v < x_e$ are not isolated.
- Do nothing until $\Phi_e(x) = y$ converges. Then split into two cases.

Case 1. y already has a neighbor in X_i.

- Declare that x_e is not isolated in G.
- Reset x_e to be the next least node that currently looks isolated.
- Restart Phase 2.

If we are wrong about x_e being not isolated, then we win R_e because $\Phi_e(x) = y$ with x isolated in G and y not isolated in X_i.

Case 2. y currently has no neighbors in X_i.

In parallel, perform two searches and act for whichever halts first.

- Does y have finite degree in G? (Assumed to be c.e. question.)
  - Promise to put y's future neighbors in X_{1-i} so y is isolated in X_i.
  - Declare that x_e is isolated in G.
  - If we are wrong about x_e being isolated, then we win R_e.
- Does y gain a new neighbor in G not yet promised to X_0 or X_1?
  - Promise to put this neighbor into X_i so y is not isolated in X_i.
  - Declare x_e is not isolated in G.
  - If we are wrong about x_e being isolated, then we win R_e.

Reset x_e to be the next currently isolated node in G and restart Phase 2.

Action. At stage s, put s into whichever of X_0 or X_1 it has been promised to, or into X_0 if it is not promised to either set.

Open questions

Does Cameron's theorem hold in REC? Can you remove our assumption that the set of nodes with finite degree is c.e.?

Is the theorem provable in RCA_0? Or in $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$?

Thank you!