Equivalents of NOTOP

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I. A history lesson

II. Revising the story

III. Under the hood

Throughout the whole talk, T is always a complete theory in a countable language.

History

Saharon Shelah

1971

Definition

A (complete, countable) stable theory T is superstable if there do not exist c and $A_0 \subseteq A_1 \subseteq A_2 \subset \dots$ with $\mathrm{tp}(c/A_{n+1})$ forking over A_n for each n.

Theorem

If T is not superstable, then the class of uncountable models of T is chaotic. (In particular, $I(T,\kappa) = 2^\kappa$ for all $\kappa > \aleph_0$.)

Henceforth, we will assume all theories are (complete) and superstable in a countable language.

T countable, superstable

Notation:

M a-saturated $\leftrightarrow$ $\mathbf{F}^a_{\aleph_0}$-saturated model $\leftrightarrow$ $\aleph_\epsilon$-saturated model means: M realizes every type in $S(\mathrm{acl}^{eq}(A))$ for every finite $A \subseteq M$.

An independent triple of models (M_0, M_1, M_2) satisfies $M_0 \preceq M_1$, $M_0 \preceq M_1$, with $M_1 \underset{M_0}{\downarrow} M_2$ An independent triple of models

(M_0, M_1, M_2) satisfies $M_0 \preceq M_1$, $M_0 \preceq M_1$, with $M_1 \underset{M_0}{\downarrow} M_2$ (forking independence!)

1981

Definition

A (countable) superstable T has NDOP if, for any independent triple (M_0, M_1, M_2) of *a*-models, any a-prime model M^* over $M_1 M_2$ is minimal over $M_1 M_2$. [If $M_1 M_2 \subseteq N \preceq M^*$, then $N = M^*$.]

Theorem (Main Gap for a-saturated models)

If T is superstable with NDOP, then every a-saturated model is a-prime and a-minimal over an independent tree $\{M_\eta : \eta \in I\}$ of a-models of size $2^{\aleph_0}$.

Theorem

If T is either unsuperstable or if T has DOP, then $I(T, \kappa) = 2^\kappa$ for all $\kappa > \aleph_0$.

1984

Definition

A (countable) superstable T has NOTOP if there **does not** exist a type $p(x, y, z)$ such that for every λ and $R \subseteq \lambda^2$, there is a model M_R and $\{a_i : i \in \lambda\} \subseteq M_R$ such that for all $(i, j) \in \lambda^2$,

$$M_R \text{ realizes } p(x, a_i, a_j) \text{ if and only if } R(i, j)$$

Definition

T is classifiable if T is countable, superstable, NDOP, NOTOP.

1989 – 2 years after Volume 2 of Classification Theory

Theorem (Main Gap)

Let T be any complete theory in a countable language.

1. *If T is not classifiable, the $I(T,\kappa) = 2^\kappa$ for all $\kappa > \aleph_0$.*
2. *if T is classifiable, then every model N is constructible and minimal over an independent tree $(M_\eta : \eta \in I)$ of countable, elementary substructures.*

Computing the 13 species of uncountable spectra starts with this.

The take-away

Historically –

NOTOP was only developed/explored in the presence of NDOP!

Will see: Countable, superstable, NOTOP theories admit structure theorems, even without NDOP.

Revisionist history: NOTOP without NDOP

Recall: $\mathrm{tp}(c/B)$ is isolated if there is some $\psi(x, b) \in \mathrm{tp}(c/B)$ such that $\psi(x, b) \vdash \mathrm{tp}(c/B)$.

Lachlan: $\mathrm{tp}(c/B)$ is ℓ-isolated if, for every $\phi(x, y)$, there is $\psi(x, b) \in \mathrm{tp}(c/B)$ such that $\psi(x, b) \vdash \mathrm{tp}_\phi(c/B)$.

A construction sequence over B $(c_\alpha : \alpha < \gamma)$ satisfies $\mathrm{tp}(c_\alpha/B \cup \{c_\beta : \beta < \alpha\})$ is isolated for all $\alpha < \gamma$.

An ℓ-construction sequence over B $(c_\alpha : \alpha < \gamma)$ satisfies $\mathrm{tp}(c_\alpha/B \cup \{c_\beta : \beta < \alpha\})$ is ℓ-isolated for all $\alpha < \gamma$.

‘Poor man’s ω-stability’

Facts:

- If T is ω-stable, then for every set B, the isolated types are dense in $S(B)$, hence constructible models exist over every set.
- If T is countable, superstable, then for every set B, the ℓ-isolated types are dense in $S(B)$, hence ℓ-constructible models exist over every set.

Independent triples of models

An Independent triple of models (M_0, M_1, M_2) satisfies $M_0 \preceq M_1$, $M_0 \preceq M_2$, and $M_1 \underset{M_0}{\perp} M_2$.

Question: For an independent triple of models, how easy is it to complete $M_1 M_2$ to a model?

- (Rarely) $M_1 M_2$ itself will be a model (e.g., theory of equality)
- Sometimes $\mathrm{acl}(M_1 M_2)$ will be a model (e.g., vector spaces or algebraically closed fields)
- (T countable, superstable) There always is an ℓ-constructible model over $M_1 M_2$.

Revisionist definitions

Suppose T is countable and superstable.

- Property A: For every independent triple of countable models (M_0, M_1, M_2), every ℓ-isolated $\mathrm{tp}(c/M_1M_2)$ is isolated.

  Fact: Property A implies every ℓ-constructible model over M_1M_2 is constructible over M_1M_2.
- Property B: For every independent triple of countable models (M_0, M_1, M_2), every ℓ-constructible model over M_1M_2 is minimal over M_1M_2.

Theorem (L-Ulrich)

For T countable, superstable,

1. *Property A is equivalent to NOTOP.*
2. *Property B implies NDOP, and $A + B$ is equivalent to T classifiable.*

Recall: T is classifiable iff every $N \models T$ is constructible and minimal over an independent tree $(M_\eta : \eta \in I)$ of countable, elementary substructures.

Theorem (L-Ulrich)

Suppose T is countable, superstable, with Property A (NOTOP).

1. *Every $N \models T$ is atomic over an independent tree $(M_\eta : \eta \in I)$ of countable, elementary substructures;*
2. *There is a constructible model $N' \preceq N$ over $\bigcup\{M_\eta : \eta \in I\}$;*
3. *If $N' \preceq N'' \preceq N$, then $N' \preceq_{\infty,\omega} N'' \preceq_{\infty,\omega} N$, i.e., all three models are back-and-forth equivalent.*

Contrast:

- If T is classifiable, then every model N has a tree $(M_\eta : \eta \in I)$ of countable, elementary substructures that determines N up to isomorphism over the tree.
- If T is countable, superstable, NOTOP, then every model N has a tree $(M_\eta : \eta \in I)$ of countable, elementary substructures that determines N up to back and forth equivalence over the tree.

'Under the hood'

Say $\overline{M} = (M_0, M_1, M_2)$, $\overline{N} = (N_0, N_1, N_2)$ are independent triples of models (of any size). Define $\overline{M} \sqsubseteq \overline{N}$ iff $M_i \preceq N_i$ for each i, $N_0 \underset{M_0}{\downarrow} M_1 M_2$, $N_1 \underset{N_0 M_1}{\downarrow} M_2$ and $N_2 \underset{N_0 M_2}{\downarrow} M_1$.

Credo: (Indep triples, $\sqsubseteq$) acts very much like (Mod(T),$\preceq$).

- If $\overline{M} \sqsubseteq \overline{N}$ then $M_1 M_2 \subseteq_{TV} N_1 N_2$;
- (ULS) For any $\overline{M}$, there is $\overline{N} \sqsupseteq \overline{M}$ consisting of a-models
- (DLS) For any $\overline{N}$ and any $X \subseteq N_1 N_2$ with $|X| \leq \kappa$, there is $\overline{M} \sqsubseteq \overline{N}$ with $X \subseteq M_1 M_2$ and $|M_1 M_2| \leq \kappa$.

Definition (Harrington)

Suppose $\overline{M} = (M_0, M_1, M_2)$ is any independent triple. We say c is V-dominated by $\overline{M}$ if, $c \underset{M_1 M_2}{\perp} N_1 N_2$ for every $\overline{N} \sqsupseteq \overline{M}$.

New: We say T has V-DI if for all c and for all $\overline{M}$, if c is V-dominated by $\overline{M}$, then $\mathrm{tp}(c/M_1 M_2)$ is isolated.

Fact: For any c and $\overline{M}$,

- If $\mathrm{tp}(c/M_1 M_2)$ is ℓ-isolated, then c is V-dominated by $\overline{M}$.
- If, in addition, each M_i is a-saturated, then the converse holds.

Will see later that V-DI is still another equivalent of NOTOP.

Theorem (L-Ulrich)

V-DI implies PMOP (existence of a constructible model over independent triples of models of any size).

Remark: The above was proved by Shelah, and reproved by Hart, both under the assumption of NDOP.

On page 619 of *Classification Theory* (1987), Shelah writes:

"Remark. Really "without the dop" is not necessary, this will be shown in a subsequent paper."

Local versions of NDOP

Fact: T has NDOP iff for all independent triples (M_0, M_1, M_2) of a-models and for all a-prime M^* over M_1M_2, every regular type $r \not\perp M^*$ is $\not\perp M_1$ or $\not\perp M_2$.

Fact: $\not\perp$ induces an equivalence relation on the set of regular types.

Let $\mathbf{P}$ be any union of $\not\perp$-classes of regular types.

Definition: T has $\mathbf{P}$-NDOP iff for all independent triples (M_0, M_1, M_2) of a-models and for all a-prime M^* over M_1M_2, every regular $r \not\perp M^*$ with $r \in \mathbf{P}$ is $\not\perp M_1$ or $\not\perp M_2$.

Thesis: NOTOP implies 'some amount of DOP', i.e., $\mathbf{P}$-NDOP for some choice of $\mathbf{P}$.

Example: A (stationary) regular type r is eni (eventually non-isolated) if there is some finite set d on which r is based and stationary with $r|d$ non-isolated. Call a $\not\perp$-class C of regular types eni if at least one $r \in C$ is eni.

Theorem (L-Shelah, 2015)

If T is ω-stable, then T has eni-NDOP iff T has NOTOP.

Thus, $\Leftarrow$ implies that NOTOP implies NDOP for all $\not\perp$-classes of eni types.

Motivating Question: Can this equivalence extend to countable, superstable theories?

A new class of regular types

Fact: $\not\perp$ is actually an equivalence relation on the (larger) class of stationary, weight one types.

Definition(Baisalov, 1990) A regular type r is $\mathbf{P}_e$ if $r \not\perp p(x, d)$ for some stationary, weight one $p(x, d)$ that is non-isolated.

Obviously, *eni* $\subseteq \mathbf{P}_e$. For T ω-stable, T has $\mathbf{P}_e$-NDOP iff T has eni-NDOP iff T has NOTOP.

But there are examples of countable, superstable $\mathbf{P}_e$-DOP theories with eni-NDOP.

Some equivalents

Theorem (L-Ulrich)

The following are equivalent for a countable, superstable T:

1. *Property A (for every independent triple of countable models $\overline{M}$, $\mathrm{tp}(c/M_1M_2)$ ℓ-isolated implies $\mathrm{tp}(c/M_1M_2)$ isolated);*
2. *T is V-DI;*
3. *T has $\mathbf{P}_e$-NDOP and countable PMOP (there exists a constructible model over every independent triple of countable models);*
4. *T has $\mathbf{P}_e$-NDOP and full PMOP (there exists a constructible model over every independent triple of models);*
5. *T has NOTOP.*

An amusing corollary

Recall: T has OTOP iff if there is a type $p(x, y, z)$ such that for every λ and $R \subseteq \lambda^2$, there is a model M_R and $\{a_i : i \in \lambda\} \subseteq M_R$ such that for all $(i, j) \in \lambda^2$,

$$M_R \text{ realizes } p(x, a_i, a_j) \text{ if and only if } R(i, j)$$

T has linear OTOP iff if there is a type $p(x, y, z)$ such that for every linear order $(L, \leq_L)$, there is a model M_L and $\{a_i : i \in L\} \subseteq M_L$ such that for all $(i, j) \in L^2$,

$$M_L \text{ realizes } p(x, a_i, a_j) \text{ if and only if } i \leq_L j$$

Corollary

For countable, superstable T, OTOP is equivalent to linear OTOP.

On adding constants

Good news: If a countable, superstable theory has OTOP, then any expansion by adding countably many constants will also have OTOP (hence, we may assume our type $p(x, y, z)$ witnessing OTOP has countably many parameters).

Danger: There is a countable, superstable theory T with OTOP, but if we add $2^{\aleph_0}$ constants naming a saturated model, then the expanded theory is categorical in all $\kappa > 2^{\aleph_0}$.

Thanks for listening!