On Winning Strategies in Σ_2^0 Games

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Winning strategies in Σ^0_1 games

Winning nodes (for player I) get ranked 0.
Inductively assign ordinals to nodes, through ω_1^{CK}, resp. ω_1^R.
Unranked nodes get labeled ∞ and are wins for II.

Winning strategies for I in Σ^0_2 games

Inductively rank nodes, through σ,

- the least Σ^1_1 reflecting ordinal,
- the least ordinal which is Π_1 gap-reflecting on admissibles,
- the closure point of Σ^1_1 monotone inductive definitions,
- the closure point of Σ_2 definitions in the μ-calculus,
- the closure point of feedback hyperarithmetic computations,
- the closure point of parallel feedback Turing computations,
- the least Σ^1_1 Ramsey ordinal,
- the least non-Gandy ordinal.

Winning strategies for II in Σ_2^0 games

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Proposition

There is a bound strictly less than σ^+ by which every Σ_2^0 game has a winning strategy.

(Non-)Gandy ordinals

Definition

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Let δ be the supremum of the order-types of the σ-recursive well-orderings of σ.

δ and winning strategies

Lemma

δ is the least upper bound of ranks of winning nodes for II in any game $\exists j\ \phi(i, j)$.

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Theorem

δ is the least ordinal β such that every Σ_2^0 game has a winning strategy **with witness** *definable over L_β.*

Refinement and question

Question: Is there a game such that II has a winning strategy strictly between σ and δ (even though the proof that it's winning is at best definable over L_δ)?

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Answer: Yes! Let II play a model of "$V = L_\sigma$"; I needs to show that II's model is incorrect, by finding an infinite descending sequence through it, or showing it fails Π_1 gap-reflection.
Question: Is there a game such that II's first winning strategy is only definable over L_δ? Or does every game have a w.s. in L_δ (albeit without a witness of such)?

Possible approaches

Goal: Build a winning strategy for II without using the ordinals through δ.
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Idea 1: Use ordinal notations, i.e. σ-recursive well-orderings of σ.
Idea 2: Work in a model M with standard ordinal part σ, because $W^M \subseteq W$ (where W is the set of winning nodes for II).

Model theory question

Consider II's L-least winning strategy for the game in which II must build a model of $V = L_{\sigma^+}$. The model II builds with this strategy has ordinal standard part σ.

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Task: Understand the model II builds in the face of I's challenges.

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