The Borel Complexity of the Class of Models of First-order Theories

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May 14, 2025

ASL Annual Meeting, New Mexico State University

Joint work with Uri Andrews, David Gonzalez, Steffen Lempp, and Dino Rossegger.
With thanks to Ali Enayat, Roman Kossak, and Albert Visser.

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Motivation

Question. How complicated is (true) arithmetic?

What could be an answer?

1. Axiomatization:
   - Incompleteness theorem
   - Undefinability of truth
2. Computing a model:
   - Tennenbaum's theorem
   - Non-standard models of PA have infinite Scott rank (Montalbán-Rossegger)
3. Distinguishing the models:
   - Separate Mod(T) from other structures.

Setting

Assume the language $\mathcal{L}$ is (at most) countable and relational. All structures will be assumed to have domain ω. Fix an enumeration $\{\varphi_i\}_{i\in\omega}$ of atomic $\mathcal{L}\cup\omega$-sentences (using ω as constants).

For any $\mathcal{L}$-structure $\mathcal{M}$, its atomic diagram can be encoded by the path $p_{\mathcal{M}} \in 2^\omega$, where $p_{\mathcal{M}}(i) = 1 \iff \mathcal{M} \vDash \varphi_i$ and 0 otherwise. Identify $\mathcal{M}$ with $p_{\mathcal{M}}$.

Throughout, let T be a (consistent) first-order $\mathcal{L}$-theory. Let $\mathrm{Mod}(T) \subseteq 2^\omega$ be the set of all countable models of T. We can analyze its complexity under the usual Polish topology on 2^ω, and it turns out to be always Borel.

Question. What is the relationship between the complexity of $\mathrm{Mod}(T)$ and that of T?

To answer that, we need to define what "complexity" means in both contexts.

The Borel Hierarchy

We will be using the Borel hierarchy ($\boldsymbol{\Sigma}^0_\alpha, \boldsymbol{\Pi}^0_\alpha, \boldsymbol{\Delta}^0_\alpha$) as our measure of complexity for $\mathrm{Mod}(T) \subseteq 2^\omega$.

Connections to computability theory:

Theorem

For any recursive ordinal α,

$$\boldsymbol{\Sigma}^0_\alpha = \bigcup_{X \in 2^\omega} \Sigma_\alpha(X), \boldsymbol{\Pi}^0_\alpha = \bigcup_{X \in 2^\omega} \Pi_\alpha(X), \boldsymbol{\Delta}^0_\alpha = \bigcup_{X \in 2^\omega} \Delta_\alpha(X).$$

Basically, we can effectively recover a Borel set from its code.

Theorem

$f : 2^\omega \to 2^\omega$ is continuous if and only if it is computable relative to some oracle $X \in 2^\omega$.

Quantifier Alternation Hierarchy

We have a similar hierarchy on the formula side.

Definition

- $\exists_1 = \{\exists \overline{x} R(\overline{x}) : R \text{ quantifier-free}\}$, $\forall_1 = \{\forall \overline{x} R(\overline{x}) : R \text{ quantifier-free}\}$
- $\exists_{n+1} = \{\exists \overline{x} R(\overline{x}) : R \in \forall_n\}$, $\forall_{n+1} = \{\forall \overline{x} R(\overline{x}) : R \in \exists_n\}$

Remark. In the context of arithmetic this is different from Σ^0_n, Π^0_n due to bounded quantifiers. However, over PA, it's safe to use either as long as $n \geq 1$ (by MRDP).

By induction, $\mathrm{Mod}(\varphi)$ is $\mathbf{\Pi}^0_n$ if $\varphi \in \forall_n$. A converse to this is formulated using infinitary logic.

Infinitary Logic

The infinitary logic $\mathcal{L}_{\omega_1\omega}$ is obtained from first-order logic by allowing countable conjunctions and disjunctions (but still referring to only finitely many free variables in a single formula). To form a similar hierarchy, we treat countable conjunctions like universal quantifiers (and similarly for countable disjunctions), resulting in the classes Π^{in}_α and $\Sigma^{\text{in}}_\alpha$.

Theorem (López-Escobar; Vaught)

If $A \subseteq 2^\omega$ is isomorphism-invariant (in particular if $A = \mathrm{Mod}(\varphi)$ for some sentence φ), then it is $\mathbf{\Pi}^0_\alpha$ iff it is $\mathrm{Mod}(\psi)$ for some $\psi \in \Pi^{in}_\alpha$.

This gives an upper bound of $\mathrm{Mod}(T)$'s complexity.

Corollary

- $\mathrm{Mod}(T)$ *is always* $\mathbf{\Pi}^0_\omega$.
- *If T is axiomatizable by $\forall_n$ (or $\exists_{n-1}$) sentences, then $\mathrm{Mod}(T)$ is $\mathbf{\Pi}^0_n$.*

The Result for Arithmetic

Theorem (Andrews, Gonzalez, Lempp, Rossegger, Z.)

Mod(TA) *and* Mod(PA) *are both* $\mathbf{\Pi}^0_\omega$*-complete.*

Idea. Given a $\mathbf{\Pi}^0_\omega$ set $P = \bigcap_n P_n$ (where P_n is $\mathbf{\Pi}^0_n$), we will build a continuous reduction f such that $f(p) \models \mathrm{TA}$ when $p \in P$ and $f(p) \not\models \mathrm{PA}$ when $p \notin P$. Thus f witnesses the $\mathbf{\Pi}^0_\omega$-hardness for both Mod(TA) and Mod(PA).

Fact

For each $n \geq 1$, $\mathrm{Th}_{\exists_n}(\mathrm{TA}) + \neg \mathrm{B}\Sigma_n$ *is consistent.*

Take T_n to be any completion of the above theory.

Construction. Starting from $n = 1$, check if $p \in P_n$. If not, proceed to make $f(p)$ a model of T_n (and stop). Otherwise, make the "$\exists_n$-fragment" of $f(p)$ satisfy $\mathrm{Th}_{\exists_n}(\mathrm{TA})$, increment n, and repeat.

General Unbounded Theories

Key. Even if we have committed to a "complete" $\exists_n$-fragment, there is still a "switch" on a higher level that we can turn off to deviate from the target theory.

This allows us to generalize the above argument to a large class of theories.

Definition (Boundedly Axiomatizable Theory)

A theory T is *boundedly axiomatizable* (*bounded* for short) if for some $n < \omega$, T has an axiomatization consisting entirely of $\forall_n$ sentences. (T is *unbounded* otherwise.)

Theorem (AGLRZ)

If T is complete, then T is unbounded $\iff$ $\mathrm{Mod}(T)$ is $\mathbf{\Pi}^0_\omega$-complete.

Remark. The assumption of completeness is necessary: there are unbounded theories with a (strictly) $\mathbf{\Delta}^0_\omega$ class of models.

The Unbounded Case

Proof. ($\Leftarrow$) If T is $\forall_n$-axiomatizable then $\mathrm{Mod}(T)$ is $\mathbf{\Pi}^0_n$, thus not $\mathbf{\Pi}^0_\omega$-hard.
($\Rightarrow$) Follow the outline, and check two things.

1. The construction is possible.
   In general, checking whether $p \in P_n$ cannot be done continuously. However, we can approximate it effectively (as we approximate the theory):

Theorem (Solovay; Knight)

Let T be a complete theory. Suppose $R \leq_T X$ is an enumeration of a Scott set $\mathcal{S}$, with functions t_n, $\Delta^0_n(X)$ uniformly in n, such that: for each n, $\lim_s t_n(s)$ is an R-index for $\mathrm{Th}_{\exists_n}(T)$; and for all s, $t_n(s)$ is an R-index for a subset of $\mathrm{Th}_{\exists_n}(T)$. Then X can compute a model $\mathcal{M} \models T$ representing $\mathcal{S}$.

   We make extensive use of the uniformity of the above theorem.
2. The desired complete theories T_n's exist.
   We just need $\mathrm{Th}_{\exists_n}(T) = \mathrm{Th}_{\exists_n}(T_n)$ and $T \neq T_n$. If this is not possible, then T is axiomatized by $\mathrm{Th}_{\exists_n}(T)$, a contradiction. □

Examples: Sequential Theories

Corollary

If T is a completion of PA, *then* $\mathrm{Mod}(T)$ *is* $\mathbf{\Pi}^0_\omega$*-complete.*

Proof.

No consistent extension of PA is bounded (Rabin). □

It turns out that there is a large class of unbounded theories:

A theory is *sequential* if, roughly speaking, it is able to encode finite sequences.
(It directly interprets *adjunctive set theory*, i.e. $\varnothing$ exists; and for all x, y, we have $x \cup \{y\}$ exists.)

Theorem (Enayat, Visser)

Every complete sequential theory in a finite language is unbounded.

Examples of sequential theories include $\mathrm{PA}, \mathrm{PA}^-, \mathrm{ZF}$, etc. and their extensions.

The Bounded Case

We can also analyze $\mathrm{Mod}(T)$ when T is bounded, even if it is incomplete:

Theorem (AGLRZ)

For any $n \in \omega$ and any theory T (not necessarily complete):

1. T *is not* $\forall_n$*-axiomatizable* $\Rightarrow \mathrm{Mod}(T)$ *is* $\mathbf{\Sigma}^0_n$*-hard. Thus,* T *is* $\forall_n$*-axiomatizable* $\iff \mathrm{Mod}(T) \in \mathbf{\Pi}^0_n$.
2. T *is not* $\exists_n$*-axiomatizable* $\Rightarrow \mathrm{Mod}(T)$ *is* $\mathbf{\Pi}^0_n$*-hard. Thus,* T *is* $\exists_n$*-axiomatizable* $\Leftarrow \mathrm{Mod}(T) \in \mathbf{\Sigma}^0_n$.
3. T *is* $\forall_n$*- but not* $\exists_n$*-axiomatizable* $\iff \mathrm{Mod}(T)$ *is* $\mathbf{\Pi}^0_n$*-complete.*
4. T *is* $\exists_n$*-axiomatizable* $\Rightarrow \mathrm{Mod}(T) \in \mathbf{\Pi}^0_{n+1}$.

(3) is immediate from (1) (2), and (4) follows from López-Escobar, so the hard work goes in (1) and (2) (which are proved similarly).

The Bounded Case

Lemma

Suppose $n \in \omega$, *and* $T^+ \neq T^-$ *are complete theories with* $\mathrm{Th}_{\exists_n}(T^-) \subseteq \mathrm{Th}_{\exists_n}(T^+)$. *Then for any* $P \in \mathbf{\Sigma}^0_n$, *there is a continuous reduction* f *such that* $f(x) \in \mathrm{Mod}(T^+)$ *if* $x \in X$, *and* $f(x) \in \mathrm{Mod}(T^-)$ *otherwise.*
In particular, $\mathrm{Mod}(T^+)$ *is* $\mathbf{\Sigma}^0_n$*-hard, and* $\mathrm{Mod}(T^-)$ *is* $\mathbf{\Pi}^0_n$*-hard.*

Remark. This is like stopping the construction for unbounded theories at level n.

Proof.

Given $p \in 2^\omega$, Feed these ingredients to Solovay's theorem:

- R: comes from a fixed oracle.
- t_k for $k < n$: output a fixed R-index of $\mathrm{Th}_{\exists_k}(T^-) = \mathrm{Th}_{\exists_k}(T^+)$.
- t_n: check whether $p \in P$ using $p^{(n-1)}$; keep outputting index of $\mathrm{Th}_{\exists_n}(T^-)$ until the Σ_n outcome (i.e. witness $p \in P$), then switch to $\mathrm{Th}_{\exists_n}(T^+)$.
- t_k for $k > n$: compute membership and then output the correct index. □

Tightness

Examples showing the $\exists_n$ result is "tight":

Remark. Using Marker's extension, one can make these work for larger values of n.

Example

Let $\mathcal{L}$ consist of just one unary relation P, and T says P is infinite and coinfinite. Then T is $\exists_1$-axiomatizable and $\aleph_0$-categorical (thus complete). $\mathrm{Mod}(T)$ is $\mathbf{\Pi}^0_2$-complete. [In fact, by our convention, $\mathrm{Mod}(T) \in \mathbf{\Sigma}^0_2 \Rightarrow \mathrm{Mod}(T) = \varnothing$.]

Example

$T = \mathrm{Th}(2 \cdot \mathbb{Q} + 1 + \mathbb{Q}, <, S)$ is axiomatizable by a single $\exists_3$ sentence and $\aleph_0$-categorical. $\mathrm{Mod}(T)$ is $\mathbf{\Sigma}^0_3$-complete.

Example

Use a 2-sorted language to combine a $\exists_2 - \mathbf{\Pi}^0_3$ example and a $\exists_3 - \mathbf{\Sigma}^0_3$ example: this gives a $\exists_3 - \mathbf{\Delta}^0_4$ (strict) example.

Connection to Infinitary Logic

While infinitary logic is more expressive than first-order logic, it does not do so more efficiently (in terms of quantifier complexity). We give a quick proof:

Theorem (Keisler 1965; Harrison-Trainor, Kretschmer 2023)

If a finitary formula φ is equivalent to a Π_n^{in} formula ψ, then φ is actually equivalent to a finitary $\forall_n$ formula.

Proof.

By compactness, it suffices to show $T = \{\varphi\}$ is $\forall_n$-axiomatizable. This is immediate as $\mathrm{Mod}(T) = \mathrm{Mod}(\psi)$ is $\mathbf{\Pi}_n^0$. □

It turns out that this is, in a sense, equivalent to our theorem. It's also interesting that Keisler used games to approximate formulas, Harrison-Trainor and Kretschmer used arithmetic forcing, while we used (complicated) priority argument, which is effective.

Further Questions

- Characterize the Wadge degrees occupied by Mod(T)?
  In particular, how do they differ from the degrees that are Scott complexities?
- Can more be said about the $\mathbf{\Pi}^0_\omega$ case when T is incomplete (and not sequential)?
- More effective analysis on oracles?

Thank you for listening!

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