Criteria for local tabularity of products of modal logics

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Part I.

Preliminaries and History

Main Definitions. Modal Logics

The **modal language** ML(A) for an alphabet of modalities A consists of **formulas** built from propositional variables PV using the Boolean connectives $\vee$, $\wedge$, $\neg$, $\ldots$ and unary connectives $\Diamond \in A$.

Definition

Logical version

A **modal logic** with alphabet of modalities A is a subset of ML(A) that contains:

- All propositional tautologies,
- $\Diamond\bot \leftrightarrow \bot$ for all $\Diamond \in A$,
- $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$ for all $\Diamond \in A$,

and is closed under Modus Ponens, Substitution, and

$$(Mon_{\Diamond}) \qquad \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi} \text{ for all } \Diamond \in A.$$

Algebraic version

A **modal algebra** for an alphabet of modalities A is a Boolean algebra endowed with unary operations $\Diamond \in A$ satisfying:

- $\Diamond 0 = 0$ for all $\Diamond \in A$,
- $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$ for all $\Diamond \in A$.

A set of terms L is a **modal logic**, if for some class $\mathcal{K}$ of modal algebras it holds that

$$\varphi \in \mathrm{L} \iff \mathcal{K} \models \varphi = 1.$$

K is the minimal unimodal logic.

Main Definitions. Kripke Semantics

We work in the standard Kripke semantics for $\mathrm{ML}(A)$:

- A **frame** is a structure $F = (X, (R_\Diamond)_{\Diamond \in A})$, where $R_\Diamond \subseteq X \times X$.
- The truth relation $(F, v), a \models \varphi$ for $a \in X$ and $v : PV \to 2^X$ is defined by:

Logical version

- $(F, v), a \models p$ iff $a \in v(p)$;
- the Boolean connectives $\bot$, $\to$ are interpreted as usual;
- $(F, v), a \models \Diamond\varphi$ iff $aR_\Diamond b$ and $(F, v), b \models \varphi$ for some point b.

Algebraic version

- $\mathrm{Alg}\, F$ is the powerset algebra of X endowed with unary operators $\Diamond_F$ for $\Diamond \in A$:

  $$\Diamond_F(Y) = {R_\Diamond}^{-1}[Y] = \{a \in X \mid \exists b \in Y\, (aR_\Diamond b)\}$$
- $\overline{v}(\varphi) \subseteq X$ is the value of the term φ in $\mathrm{Alg}\, F$ under valuation v
- $(F, v), a \models \varphi$ iff $a \in \overline{v}(\varphi)$

We say that $F \models \varphi$, if $(F, v), a \models \varphi$ for any $v : PV \to 2^X$ and any $a \in X$.

The **logic** of a class of frames $\mathcal{F}$ is $\{\varphi \in \mathrm{ML}(A) \mid \forall F \in \mathcal{F}\, (F \models \varphi)\}$. It is a modal logic for any $\mathcal{F}$.

Main Definitions. Local Tabularity and Local Finiteness

Definition

Logical version

Let L be a modal logic for an alphabet *A*. The equivalence relation $\sim_{\mathrm{L}}$ on ML(*A*) is defined by

$$\varphi \sim_L \psi \text{ iff } \varphi \leftrightarrow \psi \in \mathrm{L}.$$

A modal logic L is **locally tabular**, if for any $n \in \omega$ there are finitely many $\sim_{\mathrm{L}}$-equivalence classes of formulas in *n* variables.

Algebraic version

An algebra $\mathfrak{A}$ is **locally finite**, if any finitely generated subalgebra of $\mathfrak{A}$ is finite.

A class of algebras is **locally finite**, if it consists of locally finite algebras.

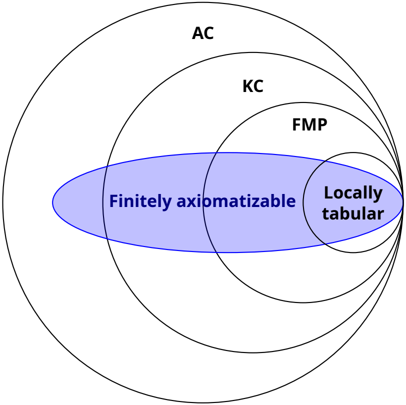
For a modal logic L, the **variety of** L**-algebras** is $\{\mathfrak{A} \mid \forall\varphi \in \mathrm{L}\,(\mathfrak{A} \models \varphi = 1)\}$.

Trivial fact: A modal logic L is locally tabular iff the variety of L-algebras is locally finite.

Hierarchy of Properties of Modal Logics

- All modal logics are **algebraically complete (AC)**.
- A modal logic is **Kripke complete (KC)**, if it is the logic of some class of frames.
- A modal logic has the **finite model property (FMP)**, if it is the logic of some class of finite frames.

All inclusions are strict:



Any finitely axiomatizable logic with FMP is decidable. In particular, any finitely axiomatizable and locally tabular logic is decidable.

Some Present Works on Locally Tabular Logics

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- Maksimova, L. *Interpolation in modal, infinite-slice logics which contain the logic* K4, 1989.
- Bezhanishvili, G. and Grigolia, R. *Locally tabular extensions of* MIPC, 1998.
- Bezhanishvili, N. *Varieties of two-dimensional cylindric algebras. Part I: Diagonal-free case*, 2002.
- Shehtman, V. *Squares of modal logics with additional connectives*, 2012.
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- Shapirovsky, I. and Shehtman, V. *Local tabularity without transitivity*, 2016.
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- Bezhanishvili, G. and Meadors, C. *Local finiteness in varieties of* MS4-*Algebras*,.
- Shapirovsky, I. *Sufficient conditions for local tabularity of a polymodal logic,* 2025.
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Towards Criterion: Clusters, Skeleton, Height

Let $F = (X, R)$ be a preordered set.

- $\sim\, = R \cap R^{-1}$ is an equivalence relation on X.
- The $\sim$-equivalence classes are called **clusters** of F.
- R induces a preorder on $X/{\sim_F}$ by

$$[a] \leq_F [b] \iff \exists c \in [a]\, \exists d \in [b]\, (cRd).$$

- The poset $\operatorname{Sk} F = (X/{\sim_F}, \leq_F)$ is called the **skeleton** of F.
- The **height** of F, denoted $h(F)$, is defined as

$$\sup \left\{ |S| \;\middle|\; S \text{ is a finite chain in } \operatorname{Sk} F \right\}.$$

Generalization. For a Kripke frame $F = (X, R)$ with arbitrary R, the height $h(F)$ is defined to be $h(X, R^*)$, where R^* is the reflexive transitive closure of R.

Theorem (Segerberg 1971)

If R is transitive, then $h(F) \leq n$ iff $F \models bh_n$, where

$$bh_0 = \bot, \quad bh_{n+1} = p_{n+1} \to \Box\,(\Diamond p_{n+1} \vee bh_n)\,;$$

$\Box\varphi$ abbreviates $\neg\Diamond\neg\varphi$.

Segerberg-Maksimova Criterion

$\mathrm{K4} = \mathrm{K} + \Diamond\Diamond p \to \Diamond p$ is the logic of all transitive frames.

The **height** of a logic $\mathrm{L} \supseteq \mathrm{K4}$ is defined as the least $n \in \omega$ such that $bh_n \in \mathrm{L}$.

Theorem (Segerberg 1971, Maksimova 1975)

A unimodal logic $\mathrm{L} \supseteq \mathrm{K4}$ *is locally tabular iff it has finite height.*

No axiomatic criterion is known for:

- The family of all unimodal logics
- Polymodal logics

Necessary Conditions of Local Tabularity: Pretransitivity, Height

- A relation R is k-**transitive**, if

$$R^{k+1} \subseteq Id \cup R \cup R^2 \cup \ldots \cup R^k.$$

- R is **pretransitive** (or **weakly transitive**), if it is pretransitive for some $k \in \omega$.
- A Kripke frame $F = (X, R)$ is k-transitive (pretransitive), if R is so.

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- A Kripke frame $F = (X, R)$ is k-transitive (pretransitive), if R is so.
- F is k-transitive iff $F \models tra_k$, where tra_k is

$$\Diamond^{k+1} p \to p \vee \Diamond p \vee \ldots \vee \Diamond^k p.$$

- A unimodal logic L is k-transitive, if $tra_k \in$ L, and pretransitive, if it is k-transitive for some k.
- Let $\Diamond^{\leq k}\varphi = \varphi \vee \Diamond\varphi \vee \ldots \vee \Diamond^k\varphi$.
- The translation $[\varphi]^k$, compatible with Boolean connectives, is given by $[\Diamond\varphi]^k = \Diamond^{\leq k}\varphi$.

Definition

The **height** h(L) of a k-transitive unimodal logic L is defined as the largest $n \in \omega$ such that $[bh_n]^k \in$ L.

Theorem (Shapirovsky & Shehtman 2016)

If L *a* 1*-finite modal logic, then* L *is pretransitive and has finite height.*

Necessary Conditions of Local Tabularity: 1-finiteness

- A modal logic L is *m***-finite**, if there are finitely many non-L-equivalent formulas built from *m* variables.
- Equivalently, L is *m*-finite, if any *m*-generated L-algebra is finite.
- By the definition, L is locally tabular iff L is *m*-finite for all $m \in \omega$.

Theorem (Maksimova 1989)

If a unimodal logic L $\supseteq$ K4 *is* 1*-finite, then it is locally tabular.*

Hence, in the transitive case:

$$\text{L is locally tabular} \iff \text{L is 1-finite} \iff \text{L has finite height.}$$

However, in general neither 1-finiteness nor finite height is sufficient for local tabularity:

- Height 1 is not sufficient for local tabularity (Byrd 1978).
  Moreover, it does not imply the 1-finiteness (Makinson 1981).
- 1-finiteness does not imply the local tabularity (Shapirovsky 2021)

Necessary Conditions for Polymodal Logics

Let $F = (X, (R_\Diamond)_{\Diamond \in A})$ be a frame for an alphabet A of modalities.

- Define $R_F = \bigcup\{R_\Diamond \mid \Diamond \in A\}$.
- The **height** and **clusters** of F are defined to be those of (X, R_F).
- F is k**-transitive**, if (X, R_F) is k-transitive. Pretransitivity is defined accordingly.
- If A is finite, then we may translate bh_n and tra_k using compound modalities to define the height and k-transitivity for polymodal logics.

Corollary

If a polymodal logic L *is* 1*-finite, then* L *is pretransitive and has finite height.*

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Part II.

Product Logics. Criteria.

Product Frames

Definition

The (modal) **product of frames** $F = (X, (R_\Diamond)_{\Diamond \in A_1})$ and $G = (Y, (S_\Diamond)_{\Diamond \in A_2})$ is defined as

$$F \times G = (X \times Y, (R^h_\Diamond)_{\Diamond \in A_1}, (R^v_\Diamond)_{\Diamond \in A_2}),$$
$$R^h_\Diamond = \{((a, b), (c, b)) \mid a, c \in X, b \in Y, aRc\}, \quad \Diamond \in A_1$$
$$R^v_\Diamond = \{((a, b), (a, d)) \mid a \in X, b, d \in Y, bSd\}, \quad \Diamond \in A_2.$$

We say that the relation R^h is **horizontal**, and R^v is **vertical**.
For classes of frames $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{F} \times \mathcal{G} = \{F \times G \mid F \in \mathcal{F} \text{ and } G \in \mathcal{G}\}$.

Definition

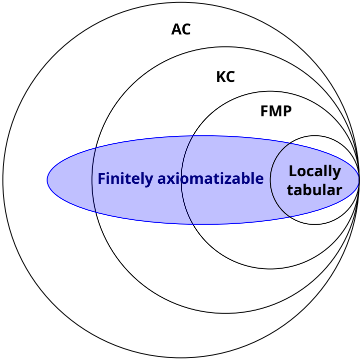
The **product of logics** L_1 and L_2 is

$$\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{Log}\{F \times G \mid F \models \mathrm{L}_1 \text{ and } G \models \mathrm{L}_2\}$$

Classic Problem: Find product logics with the finite model property.

Our goal: Find locally tabular product logics.

Neither FMP nor LT is preserved in products!



Conditions for LT of products: Bounded Cluster Property

- A class of frames $\mathcal{F}$ has the **bounded cluster property** BC_m, if

$$\sup\{|C| \mid C \text{ is a cluster in some } F \in \mathcal{F}\} \leq m.$$

- If $\mathcal{F}$ is a class of k-transitive frames of finite height, then there exists a modal formula $bc_{m,k}$ such that

$$\mathcal{F} \text{ has } \mathrm{BC}_m \iff \mathcal{F} \models bc_{m,k}.$$

Conditions for LT of products: Reducible Path Property

Let $F = (X, R)$ be a unimodal frame.

- F has the **reducible path property** RP_m, if any path between a pair of points in F contains a subpath of length at most m between the same points.
- This property is definable by the modal formula rp_m:

$$p_0 \wedge \Diamond (p_1 \wedge \Diamond (p_2 \wedge \ldots \wedge \Diamond p_{m+1}) \ldots) \to \bigvee_{i<j\leq m+1} \Diamond^i (p_i \wedge p_j) \vee \bigvee_{i<j\leq m} \Diamond^i (p_i \wedge \Diamond p_{j+1}).$$

A frame $F = (X, (R_\Diamond)_{\Diamond \in A})$ has RP_m, if $\left(X, \bigcup_{\Diamond \in A} R_\Diamond\right)$ does.

Theorem (Shapirovsky & Shehtman 2016)

If a class of frames $\mathcal{F}$ has a locally tabular logic, then $\mathcal{F}$ has RP_m *for some $m \in \omega$.*

Main Result

Theorem (Shapirovsky & V.S.)

Semantic criterion

Let $\mathcal{F}$ and $\mathcal{G}$ be nonempty classes of frames. The following are equivalent:

1. $\mathrm{Log}(\mathcal{F} \times \mathcal{G})$ *is locally tabular.*
2. $\mathrm{Log}\,\mathcal{F}$ *and* $\mathrm{Log}\,\mathcal{G}$ *are locally tabular and at least one of* $\mathcal{F}$, $\mathcal{G}$ *has the bounded cluster property.*
3. $\mathrm{Log}\,\mathcal{F}$ *and* $\mathrm{Log}\,\mathcal{G}$ *are locally tabular and* $\mathcal{F} \times \mathcal{G}$ *has the reducible path property.*
4. $\mathrm{Log}\,\mathcal{F}$ *and* $\mathrm{Log}\,\mathcal{G}$ *are locally tabular and* $\mathrm{Log}(\mathcal{F} \times \mathcal{G})$ *is 1-finite.*

The bounded cluster and reducible path properties are modally definable, so we have:

Syntactic criterion

Let L_1 *and* L_2 *be Kripke complete consistent modal logics. The following are equivalent:*

1. $L_1 \times L_2$ *is locally tabular.*
2. L_1 *and* L_2 *are locally tabular and at least one of them contains a bounded cluster property formula.*
3. L_1 *and* L_2 *are locally tabular, and* $L_1 \times L_2$ *contains a reducible path property formula.*

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Part III.

Variations.

Applications

Example

The following logics are locally tabular:

1. $L_1 \times L_2$, where $L_1 \vdash \Box_A^m \bot$ and L_2 is locally tabular — this is a generalization of (Shehtman 2012);
2. $(\mathrm{GL} + bh_n) \times \mathrm{L}$ where L is locally tabular and $n \in \omega$
(GL is the logic of all strict partial orders without infinite ascending chains);
3. $(\mathrm{Grz} + bh_n) \times \mathrm{L}$ where L is locally tabular and $n \in \omega$
(Grz is the logic of all non-strict partial orders without infinite ascending chains);
4. All extensions of the logics above.

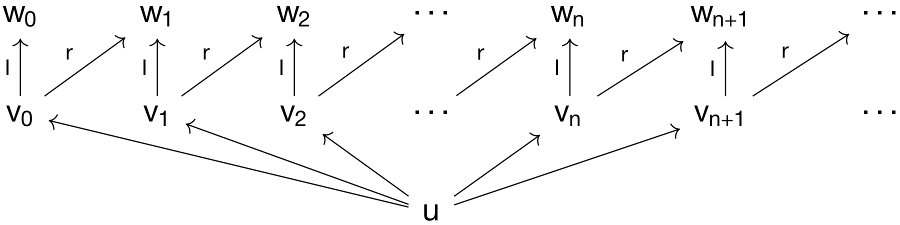
Product Finite Model Property

A logic L has the **product finite model property**, if L is the logic of a class of finite product frames.

Theorem (Shapirovsky & V.S.)

Local tabularity is not sufficient for the product finite model property.

Let L denote the logic of this frame:



Then L $\times$ S5 is locally tabular but lacks the product fmp.

Prelocal Tabularity

A logic L is **prelocally tabular**, if L is not locally tabular and all other modal logics that contain L are locally tabular.

Theorem (Maksimova 1975)

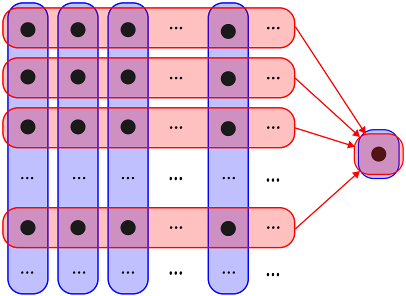
The logic Grz3 *of finite linear orders is the only prelocally tabular logic above* S4*.*

Theorem (N. Bezhanishvili 2002)

The logic S5 $\times$ S5 *is prelocally tabular, where* S5 *is the logic of all equivalence relations.*

Theorem (Shapirovsky & V.S.)

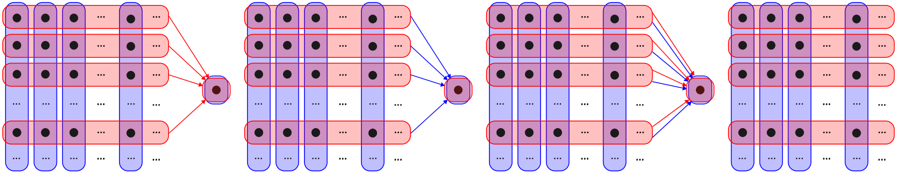
The logic of this frame is another prelocally tabular extension of S4 $\times$ S4*:*



Further Results: Prelocal Tabularity Above $S4[h] \times S4[l]$

Theorem

There are exactly four prelocally tabular logics above $S4[h] \times S4[l]$*, namely the logics of the following frames:*



Theorem

Every logic above $S4[h] \times S4[l]$ *is locally tabular or contained in a prelocally tabular logic.*

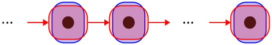
Theorem

Every logic above $S4[h] \times S4[l]$ *is locally tabular iff it contains a reducible path property formula.*

Even a stronger statement is true:

Theorem

Let $\mathcal{F}$ *and* $\mathcal{G}$ *be some classes of preorders with Noetherian skeletons, and* $L = \mathrm{Log}(\mathcal{F} \times \mathcal{G})$*. Then every normal extension of* L *is locally tabular or contained in one of the four logics above,* $\mathrm{Grz3} \times \mathrm{Triv}$*, or* $\mathrm{Triv} \times \mathrm{Grz3}$.

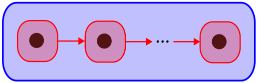


Prelocal Tabularity Above S4 $\times$ S4

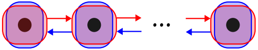
Theorem

The following logics are prelocally tabular:

- $\mathrm{Grz3U}^{\downarrow} = \mathrm{Log}\{(m, \leq, m \times m) \mid m < \omega\}$



- $\mathrm{LinTGrz} = \mathrm{Log}\{(m, \leq, \geq) \mid m < \omega\}$



LinT is the logic of all frames (X, R, R^{-1}), where R is a linear preorder.

Theorem

Every logic above LinT *is locally tabular or contained in* LinTGrz*.*

Open Problem

Classify all prelocally tabular logics above S4 $\times$ S4.

Open Problem

Is every non-locally tabular extension of S4 $\times$ S4 contained in a prelocally tabular logic?

Prelocal Tabularity Above S4 × S4

The story continues...