Initial Tukey Structure Below a Stable Ordered-Union Ultrafilter

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Ultrafilters

Definition

Let I be an infinite set and assume $\mathcal{B} \subseteq \mathcal{P}(I)$. We define

$$\langle \mathcal{B} \rangle = \{A \subseteq I : (\exists X_0, \dots, X_{n-1} \in \mathcal{B}) \bigcap_{i<n} X_i \subseteq A, \text{ for some } n \in \omega\}.$$

If $\mathcal{U}$ is an ultrafilter on I and $\langle \mathcal{B} \rangle = \mathcal{U}$, then we call $\mathcal{B}$ *a filter basis for* $\mathcal{U}$.

In our setting, I will always be countable and all of the ultrafilters on I will be nonprincipal.

Definition

Let $\mathcal{U}$ be an ultrafilter on I, and let $\mathcal{U}_i$ be an ultrafilter on J, for all $i \in I$. We define the following ultrafilter on $I \times J$:

$$\lim_{i \to \mathcal{U}} \mathcal{U}_i = \{X \subseteq I \times J : \{i :\in I : (X)_i \in \mathcal{U}_i\} \in \mathcal{U}\}.$$

When all $\mathcal{U}_i$'s are the same ultrafilter $\mathcal{V}$, we simply write $\mathcal{U} \cdot \mathcal{V}$.

Preorders on the Class of Ultrafilters

Let I, J be countable sets and let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on I and J, respectively.

Definition

1. For a function $f : I \to J$, we define the *RK-image of $\mathcal{U}$ under f* as $f^*(\mathcal{U}) = \langle\{f''X : X \in \mathcal{U}\}\rangle$. Note that $f^*(\mathcal{U})$ is an ultrafilter on J.
2. We say *$\mathcal{V}$ is RK-reducible to $\mathcal{U}$* (or, *$\mathcal{U}$ is RK-above $\mathcal{V}$*) and write $\mathcal{V} \leq_{RK} \mathcal{U}$, if there is a function $f : I \to J$ such that $f^*(\mathcal{U}) = \mathcal{V}$.
3. We say *$\mathcal{U}$ and $\mathcal{V}$ are RK-equivalent*, and write $\mathcal{U} \equiv_{RK} \mathcal{V}$, if both $\mathcal{V} \leq_{RK} \mathcal{U}$ and $\mathcal{U} \leq_{RK} \mathcal{V}$.

Fact

$\mathcal{U} \equiv_{RK} \mathcal{V}$ if and only if there is a bijection $f : I \to J$ with $f^*(\mathcal{U}) = \mathcal{V}$ if and only if there is an injection $f : I \to J$ with $f^*(\mathcal{U}) = \mathcal{V}$.

Remark

RK-equivalence is an isomorphism notion for ultrafilters.

Preorders on the Class of Ultrafilters

Recall that an ultrafilter $\mathcal{U}$ on ω is called *Ramsey*, if for all $c : [\omega]^2 \to 2$, there is $X \in \mathcal{U}$ such that $c \restriction [X]^2$ is constant.

Theorem (Blass, 1973 [Bla73])

Ramsey ultrafilters are RK-minimal. That is, if $\mathcal{U}$ is Ramsey and $\mathcal{V} \leq_{RK} \mathcal{U}$, then either $\mathcal{V}$ is principal or $\mathcal{V} \equiv_{RK} \mathcal{U}$.

Preorders on the Class of Ultrafilters

We now define the notion of *Tukey order* on the class of partial orders, introduced by Tukey ([Tuk40]) when he studied convergence in general topological spaces.

Definition

Let $(\mathbb{P}, \leq_{\mathbb{P}})$, $(\mathbb{Q}, \leq_{\mathbb{Q}})$ be partial orders.

1. A set $\mathcal{C} \subseteq \mathbb{P}$ is called *cofinal*, if for all $p \in \mathbb{P}$, there is $q \in \mathcal{C}$ with $p \leq_{\mathbb{P}} q$.
2. A map $f : \mathbb{P} \to \mathbb{Q}$ is called *cofinal*, if for every cofinal $\mathcal{C} \subseteq \mathbb{P}$, $f''\mathcal{C}$ is cofinal in $\mathbb{Q}$.
3. A map $f : \mathbb{P} \to \mathbb{Q}$ is called *monotone*, if for every $x, y \in \mathbb{P}$ with $x \leq_{\mathbb{P}} y$, we have $f(x) \leq_{\mathbb{Q}} f(y)$.
4. We say $(\mathbb{Q}, \leq_{\mathbb{Q}})$ *is Tukey reducible to* $(\mathbb{P}, \leq_{\mathbb{P}})$, and write $(\mathbb{Q}, \leq_{\mathbb{Q}}) \leq_T (\mathbb{P}, \leq_{\mathbb{P}})$, if there is a cofinal map $f : \mathbb{P} \to \mathbb{Q}$.

Preorders on the Class of Ultrafilters

We will restrict our attention to the class of directed partial orders of the form $(\mathbb{P}, \leq_{\mathbb{P}}) = (\mathcal{U}, \supseteq)$, where $\mathcal{U}$ is an ultrafilter on a countable set.

Definition

Let $\mathcal{U}$ be an ultrafilter on I and let $\mathcal{V}$ be an ultrafilter on J.

1. We write $\mathcal{V} \leq_T \mathcal{U}$, if $(\mathcal{V}, \supseteq) \leq_T (\mathcal{U}, \supseteq)$.
2. We say *$\mathcal{U}$ and $\mathcal{V}$ are Tukey equivalent*, and write $\mathcal{U} \equiv_T \mathcal{V}$, if both $\mathcal{V} \leq_T \mathcal{U}$ and $\mathcal{U} \leq_T \mathcal{V}$.

Isbell ([Isb65]) and Juhász ([Juh67]) were the first to consider the class of ultrafilters under the Tukey order. This study was revived by Milovich ([Mil08]), and later examined in detail by Dobrinen and Todorcevic ([DT11]).

Fact

1. If $\mathcal{V} \leq_T \mathcal{U}$, then this can be witnessed by a monotone cofinal map $f : \mathcal{U} \to \mathcal{V}$.
2. If $\mathcal{V} \leq_{RK} \mathcal{U}$, then $\mathcal{V} \leq_T \mathcal{U}$.

History

Todorcevic proved that there is an analogue of the result of Blass about the RK-minimality of Ramsey ultrafilters.

Theorem (Raghavan and Todorcevic, 2012 [RT12])

Ramsey ultrafliters are Tukey minimal. That is, if $\mathcal{U}$ is Ramsey and $\mathcal{V} \leq_T \mathcal{U}$, then either $\mathcal{V}$ is principal or $\mathcal{V} \equiv_T \mathcal{U}$.

An important piece of his argument used Pudlák and Rödl's canonization theorem for functions on fronts of the Ellentuck space. This brings us to an important theme that frequently shows up in initial Tukey structure work, which is the usage of "canonization theorems" for topological Ramsey spaces.

Fronts

Definition

Let $X \in [\omega]^\omega$ and assume $\mathcal{F} \subseteq [X]^{<\omega} = \{s \in [\omega]^{<\omega} : s \subseteq X\}$. We call $\mathcal{F}$ a *front on* X if the following hold:

1. $s \not\sqsubset t$ for all $s, t \in \mathcal{F}$.
2. For all $Y \in [X]^\omega$, there is (unique) $s \in \mathcal{F}$ with $s \sqsubset Y$.

Examples of fronts: $[\omega]^n$ on ω for all $n \in \omega$.

An example of "infinite rank": $\mathcal{F} = \{s \in [\omega]^{<\omega} : |s| = \min(s) + 1\}$ on ω.

For $Y \in [X]^\omega$, define $\mathcal{F}|Y = \{s \in \mathcal{F} : s \subseteq Y\}$. One of the important reasons we are interested in fronts is the following Ramsey property:

Theorem (Nash-Williams, 1965 [NW65])

Let $\mathcal{F}$ be a front on $X \in [\omega]^\omega$. Suppose $\mathcal{F} = \mathcal{F}_0 \sqcup \mathcal{F}_1$. Then there is an infinite subset $Y \subseteq X$, and some $i \in 2$ with $\mathcal{F}|Y \subseteq \mathcal{F}_i$.

Pudlák-Rödl Canonization Theorem

Theorem (Pudlák and Rödl, 1982 [PR82])

Let $\mathcal{F}$ be a front on X. Suppose $f : \mathcal{F} \to \omega$ is an arbitrary function. Then there is some $Y \in [X]^{\omega}$, and a function $\varphi : \mathcal{F}|Y \to [\omega]^{<\omega}$ with the following properties:

1. *For all $s \in \mathcal{F}|Y$, $\varphi(s) \subseteq s$.*
2. *For all $s, t \in \mathcal{F}|Y$, $f(s) = f(t)$ if and only if $\varphi(s) = \varphi(t)$.*

Remark

This can equivalently be stated for equivalence relations on $\mathcal{F}$, instead of functions.

Further History

Laflamme ([Laf89]) defined partial orders $\mathbb{P}_\alpha$ for $1 \leq \alpha < \omega_1$, and proved that the nonprincipal ultrafilters RK-reducible to the forced ultrafilter $\mathcal{U}_\alpha$ form a descending chain of order type $\alpha + 1$ under $\leq_{RK}$.

Later, Dobrinen and Todorcevic ([DT14] and [DT15]) generalized this work to construct topological Ramsey spaces $\mathcal{R}_\alpha$ densely embedded in $\mathbb{P}_\alpha$, for $1 \leq \alpha < \omega_1$. They proved canonization theorems for fronts of these spaces to completely classify all isomorphism classes of ultrafilters which are Tukey reducible to $\mathcal{U}_\alpha$, for each $1 \leq \alpha < \omega_1$. Moreover, they proved that the Tukey types of nonprincipal ultrafilters Tukey reducible to $\mathcal{U}_\alpha$ form a descending chain of order type $\alpha + 1$, as well.

Further work on initial Tukey structure problems includes [Dob16a], [Dob16b], and [DMT17]. More detailed information can be found in [Dob21].

The Space $\mathrm{FIN}^{[\infty]}$

FIN denotes the set of nonempty finite subsets of ω. For $s, t \in \mathrm{FIN}$, we write $s <_b t$ if $\max(s) < \min(t)$.

A sequence $X : \mathrm{dom}(X) \to \mathrm{FIN}$, where $\mathrm{dom}(X) \in \omega \cup \{\omega\}$, is called a *block sequence*, if $X(i) <_b X(i+1)$, for all $i+1 \in \mathrm{dom}(X)$. $\mathrm{FIN}^{[\infty]}$ denotes the set of all infinite block sequences. $\mathrm{FIN}^{[n]}$ denotes the set of finite block sequences of length n, where $n \in \omega$. Finally, $\mathrm{FIN}^{[<\infty]} = \bigcup_{n \in \omega} \mathrm{FIN}^{[n]}$.

For a block sequence X, we define $[X] = \{\bigcup_{i \in I} X(i) : I \in \mathrm{FIN}\}$. For block sequences X and Y, we write $X \leq Y$ to mean $[X] \subseteq [Y]$.
The reason we are interested in this space is the famous theorem of Hindman and its later generalization by Taylor:

Theorem

1. *Let $c : \mathrm{FIN} \to 2$ be a coloring. Then there is some $X \in \mathrm{FIN}^{[\infty]}$ such that $c \restriction [X]$ is constant. (Hindman, 1974 [Hin74])*
2. *Let $c : \mathrm{FIN}^{[n]} \to 2$ be a coloring, where $n \in \omega$. Then there is some $X \in \mathrm{FIN}^{[\infty]}$ such that $c \restriction [X]^n$ is constant. (Taylor, 1976 [Tay76])*

Ultrafilters on FIN

Definition (Blass, 1987 [Bla87])

Let $\mathcal{U}$ be an ultrafilter on FIN.

1. $\mathcal{U}$ is called *ordered-union*, if it has a basis $\mathcal{B}$ consisting of sets of the form $[X]$, where $X \in \mathrm{FIN}^{[\infty]}$.
2. $\mathcal{U}$ is called *stable ordered-union*, if it is ordered-union and for all colorings $c : \mathrm{FIN}^{[2]} \to 2$, there is some $X \in \mathrm{FIN}^{[\infty]}$ with $[X] \in \mathcal{U}$, such that $c \restriction [X]^2$ is constant.

For a stable ordered-union $\mathcal{U}$, define $\mathcal{U}_{\mathrm{min}} := \mathrm{min}^*(\mathcal{U})$, $\mathcal{U}_{\mathrm{max}} := \mathrm{max}^*(\mathcal{U})$, and $\mathcal{U}_{\mathrm{minmax}} := \mathrm{minmax}^*(\mathcal{U})$, where $\mathrm{minmax} : \mathrm{FIN} \to \omega^2$ is defined by $\mathrm{minmax}(s) = (\mathrm{min}(s), \mathrm{max}(s))$, for all $s \in \mathrm{FIN}$.

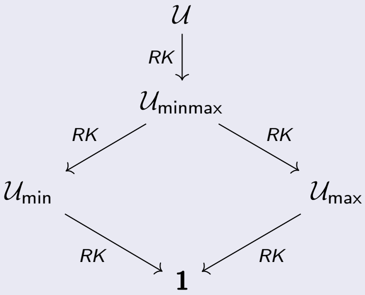
Theorem

1. *$\mathcal{U}_{\mathrm{min}}$ and $\mathcal{U}_{\mathrm{max}}$ are RK-incomparable, and $\mathcal{U} >_{RK} \mathcal{U}_{\mathrm{minmax}}$. ([Bla87])*
2. *$\mathcal{U}_{\mathrm{min}}$ and $\mathcal{U}_{\mathrm{max}}$ Tukey-incomparable. Moreover, assuming CH, there is a stable ordered-union $\mathcal{U}$ such that $\mathcal{U} >_T \mathcal{U}_{\mathrm{minmax}}$. ([DT11])*

Initial Structure Results for Stable Ordered-Union Ultrafilters

Theorem (Blass, 1987 [Bla87])

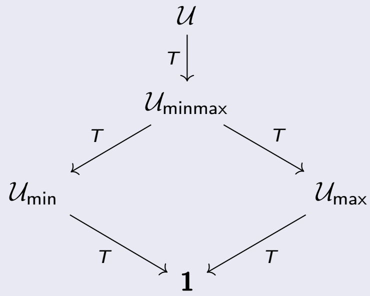
Let $\mathcal{U}$ be a stable ordered-union ultrafilter. If $\mathcal{V} \leq_{RK} \mathcal{U}$ is a nonprincipal ultrafilter, then either $\mathcal{V} \equiv_{RK} \mathcal{U}$, or $\mathcal{V} \equiv_{RK} \mathcal{U}_{\text{minmax}}$, or $\mathcal{V} \equiv_{RK} \mathcal{U}_{\text{min}}$, or $\mathcal{V} \equiv_{RK} \mathcal{U}_{\text{max}}$. This results in the following picture, where $\mathbf{1}$ *denotes a principal ultrafilter:*



Initial Structure Results for Stable Ordered-Union Ultrafilters, Main Theorem

Theorem (Ö., 2024 [Öz24])

Let $\mathcal{U}$ be a stable ordered-union ultrafilter. If $\mathcal{V} \leq_T \mathcal{U}$ is a nonprincipal ultrafilter, then either $\mathcal{V} \equiv_T \mathcal{U}$, or $\mathcal{V} \equiv_T \mathcal{U}_{\text{minmax}}$, or $\mathcal{V} \equiv_T \mathcal{U}_{\text{min}}$, or $\mathcal{V} \equiv_T \mathcal{U}_{\text{max}}$. Furthermore, $\mathcal{U} >_T \mathcal{U}_{\text{minmax}}$. This results in the following picture:



Canonization Theorems for $\mathrm{FIN}^{[\infty]}$

A key step in the proof utilized canonization theorems for $\mathrm{FIN}^{[\infty]}$.

Theorem (Taylor, 1976 [Tay76])

Suppose $f : [X] \to \omega$ is an arbitrary function, where $X \in \mathrm{FIN}^{[\infty]}$. Let $\tilde{0} : \mathrm{FIN} \to \omega$ denote the constant 0 map. Then there is some $Y \leq X$, $Y \in \mathrm{FIN}^{[\infty]}$, and $c \in \{\tilde{0}, \min, \max, \mathrm{minmax}, \mathrm{id}\}$ such that

$$f(s) = f(t) \text{ if and only if } c(s) = c(t), \text{ for all } s, t \in [Y].$$

This was later generalized by Lefmann:

Theorem (Lefmann, 1996 [Lef96])

Let k be a positive integer, and suppose $f : [X]^k \to \omega$ is an arbitrary function, where $X \in \mathrm{FIN}^{[\infty]}$. Then, there is $k_0 \leq k$, $Y \leq X$ with $Y \in \mathrm{FIN}^{[\infty]}$, a sequence $\{A_i\}_{i<k_0}$ of subsets of k with $A_i <_b A_{i+1}$ for all $i < k_0 - 1$, and corresponding functions $c_i \in \{\min, \max, \mathrm{minmax}, \mathrm{id}\}$ for $i < k_0$ such that:

1. *If $|A_i| > 1$, then $f_i = \mathrm{id}$, for all $i < k_0$.*
2. *$f(s_0, \ldots, s_{k-1}) = f(t_0, \ldots, t_{k-1})$ if and only if $(c_i(\bigcup_{j \in A_i} s_j))_{i<k_0} = (c_i(\bigcup_{j \in A_i} t_j))_{i<k_0}$.*

Canonization Theorems for $\mathrm{FIN}^{[\infty]}$

Examples of canonical functions:

1. For $\mathcal{F} = \mathrm{FIN}^{[3]}$, $c : \mathrm{FIN}^{[3]} \to \mathrm{FIN}$ defined by $(s_0, s_1, s_2) \mapsto s_0 \cup s_1 \cup s_2$.
2. For $\mathcal{F} = \mathrm{FIN}^{[6]}$, $c : \mathrm{FIN}^{[6]} \to \mathrm{FIN}^{[3]}$ defined by $(s_0, s_1, s_2, s_3, s_4, s_5) \mapsto (s_0 \cup s_1, \max(s_3), \mathrm{minmax}(s_4), s_5)$.

Klein and Spinas further generalized this result to canonize arbitrary Borel maps $f : \mathrm{FIN}^{[\infty]} \to \mathbb{R}$ ([KS05]).

This result had to be modified to get a canonization theorem for fronts that will work for the initial Tukey structure result ([Öz24]). Roughly speaking, this canonization theorem has a form very similar to Lefmann's result; only difference is that we don't have "uniform" coordinates that we project onto now as the front $\mathcal{F}$ may have infinite rank.

Canonization Theorems for $\mathrm{FIN}^{[\infty]}$

For a front $\mathcal{F}$, we define $\hat{\mathcal{F}} = \{b \in \mathrm{FIN}^{[<\infty]} : b \sqsubseteq a, \text{ for some } a \in \mathcal{F}\}$.

Theorem (Ö., 2024 [Öz24])

Let $\mathcal{F}$ be a front on some $X \in \mathrm{FIN}^{[\infty]}$, and let $g : \mathcal{F} \to \omega$ be a function. Then there is $Y \leq X$, and $\gamma : (\hat{\mathcal{F}} \setminus \mathcal{F})|Y \to \{\varnothing, \mathrm{min}, \mathrm{max}, \mathrm{minmax}, \mathrm{sss}, \mathrm{vss}\}$ such that for each $a, b \in \mathcal{F}|Y$,

$$g(a) = g(b) \text{ if and only if } \Gamma_\gamma(a) = \Gamma_\gamma(b).$$

Proof Sketch

Fix a stable ordered-union $\mathcal{U}$, and a generating set $\mathcal{B}$ for $\mathcal{U}$ consisting of infinite block sequences.

We need a few more tools to start the sketch of the proof of the main theorem. The following is a special case of a general theorem from [DMT17].

Theorem (Dobrinen, Mijares and Trujillo, 2017 [DMT17])

Whenever $\mathcal{V}$ is a nonprincipal ultrafilter on ω and $f : \mathcal{B} \to \mathcal{V}$ is monotone and cofinal, there is $X \in \mathcal{B}$ and a finitary function $\hat{f} : \mathrm{FIN}^{[<\infty]} \to [\omega]^{<\omega}$ such that $f(Y) = \bigcup_{k \in \omega} \hat{f}(Y \restriction k)$, for all $Y \leq X$ with $Y \in \mathcal{B}$.

Theorem

1. $\mathcal{U}^2_{\min} \equiv_T \mathcal{U}_{\min}$ *and* $\mathcal{U}^2_{\max} \equiv_T \mathcal{U}_{\max}$ *(Dobrinen and Todorcevic, 2011 [DT11]).*
2. $\mathcal{U}^2 \equiv_T \mathcal{U}$ *and* $\mathcal{U}^2_{\mathrm{minmax}} \equiv_T \mathcal{U}_{\mathrm{minmax}}$ *(Benhamou and Dobrinen, 2023 [BD23]).*

Definition

For a front $\mathcal{F}$ on $\mathrm{FIN}^{[\infty]}$, we define $\mathcal{U} \restriction \mathcal{F} = \langle \mathcal{F} | X : X \in \mathcal{B} \rangle$.

Proof Sketch

Proof sketch of the main theorem.

1. Assume $\mathcal{U} \geq_T \mathcal{V}$, where $\mathcal{V}$ is nonprincipal. Take monotone cofinal $f : \mathcal{B} \to \mathcal{V}$ witnessing this.
2. By the result in [DMT17], we may assume that f is generated by the finitary map $\hat{f}$.
3. Define $\mathcal{F} = \{a \in \mathrm{FIN}^{[<\infty]} : a \text{ is minimal with } \hat{f}(a) \neq \varnothing\}$. Then $\mathcal{F}$ is a front. Moreover, define $g : \mathcal{F} \to \omega$ by $g(a) = \min(\hat{f}(a))$. One can prove that $g^*(\mathcal{U} \upharpoonright \mathcal{F}) = \mathcal{V}$.
4. Canonize g to get a canonical map c with $c(\mathcal{U} \upharpoonright \mathcal{F}) \equiv_{RK} g(\mathcal{U} \upharpoonright \mathcal{F}) = \mathcal{V}$.
5. Since c is canonical, $\mathcal{V} \equiv_{RK} c(\mathcal{U} \upharpoonright \mathcal{F})$ turns out to be a countable Fubini iterate of $\mathcal{U}, \mathcal{U}_{\mathrm{minmax}}, \mathcal{U}_{\mathrm{min}}$, and $\mathcal{U}_{\mathrm{max}}$.
6. By idempotency, it follows that $\mathcal{V}$ is Tukey equivalent to one of these, and we are done.

□

Current Work

For $k \in \omega$, define FIN_k to be the set of functions $s : \omega \to k+1$ which are nonzero on a finite set, denoted supp(f), and $k \in$ ran(f). For $s, t \in \mathrm{FIN}_k$, we write $s <_b t$ if max(supp(s)) $<$ min(supp(t)), and define $\mathrm{FIN}_k^{[\infty]}$ analogously.

Lopez-Abad ([LA07]) canonized functions $f : \mathrm{FIN}_k \to \omega$.

In works: Prove a canonization theorem for functions on fronts of $\mathrm{FIN}_2^{[\infty]}$, and classify the initial Tukey structure below the corresponding ultrafilter.

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