The Black Swamp Problem Book

1. Embed every totally ordered group
5/18/78 in a divisible totally ordered group.

variation: Embed every totally orderable group in a totally orderable group which is divisible.

(Seems to have been first mentioned by B.H. Neumann ~ 1950)

1'. Of the same ilk & time (by same questioner):
Does there exist a totally ordered group all of whose strictly positive elements are conjugate? If so, can every totally ordered group be embedded in such?

2. Does every torsion-free abelian group admit an archimedean lattice order?
5/18/78

(Weinberg? ~'62)

Remark: This is related to the question of the existence of torsion-free abelian groups of cardinal $> 2^{\aleph_0}$ in which each pure subgroup is indecomposable. (E.(w.)

The question in the remark has been answered in the negative by P. Griffith [Proc. Amer. Math. Soc. 18 (1967), 738-42]. (L. Fuchs)

3. (The stranded primes). If every
5/18/78 proper prime convex
l-subgroup an an l-group G
exceeds a unique minimal
prime convex l-subgroup, is
G normal valued?
(Conrad, spring '75)

False if and only if there is an o-2 transitive (G, Ω)
having the property.

No. George Bergman (U. Calif, Berkeley)
1979.

4. If each element of an l-group
5/18/78 G has a normal value, is G
normal valued?

(Paris 67)

May 5, 1986: No (attributed to Rick Ball)

$G =$ (finitely) piecewise linear o-permutations of $\mathbb{R}$ of bounded support. If $g \in G$ and $r =$ l.u.b supp g then $V = \{f \in G \mid f$ fixes an interval $[s,r], s<r\}$ is a value of g normal in its cover ($= G_r$). But G is o-2-transitive on $\mathbb{R}$, so not normal valued.

5. Can the carrier (ie, the totally
[illegible]/5/78 ordered set) of every o-group
be the carrier of an <u>abelian</u>
o-group?

(Asked of C. Holland by R. Baer ~ 1961)

<u>No</u>. $\exists$ a totally ordered set $(G, \leq)$ such
that
(1) For a certain group operation $\cdot$ on G,
$(G, \cdot, \leq)$ is an o-group, and
(2) If $*$ is any group operation on G
such that $(G, *, \leq)$ is an o-group,
then $(G, *)$ satisfies <u>no</u> equational
group law (except those satisfied by <u>all</u> groups)

C. Holland, A. Mekler, S. Shelah, 1983

6. Is every residually lattice-orderable group lattice-orderable?

5/18/78

June 1980 No

Let G_n be the group generated by a, b, c with $ab = ba$, $c^{-1}ac = b$, $c^{-1}bc = a$ for each $n = 1, 2, \ldots$ (the Scrimger 2-group).

Let G be the subgroup of $\prod_{n=1}^{\infty} G_n$ generated by the restricted product and the two diagonals on ac and bc, respectively. Since $(ac)^2 = (bc)^2$, the two diagonals have the same square. But a short computation shows they are not conjugate in G. Hence, by the Byrd-Lloyd-Teller-Mena condition, G is not lattice orderable.

Charles Holland

7. If $\mathcal{G}$ is an o-2-transitive l-group with more than one element, is every l-group embeddable in an ultrapower of $\mathcal{G}$?

J McNulty
Oct 78

No! The first order ~~statement~~

"$\forall x, y,\ x^2 = y^2 \Rightarrow x = y$" is true in some o-2-transitive group (see below), and if the answer to this question were "yes", the statement would have to hold in all o-2-transitive groups, which obviously is not the case.

An o-2-transitive group with unique square root extraction is the l-subgroup of $A(\mathbb{R})$ consisting of the piecewise linear functions having bounded support.

Steve McCleary
Nov 17, 1978

8. If $\mathcal{V} \supsetneqq \mathcal{A}$ is a representable $\underbrace{\text{variety}}_{}$ of ℓ-groups, does $\mathcal{V}$ contain a non-abelian soluble ℓ-group?

(Rhemtulla 1977).

No, because of #23.

9. Can normal-valuedness be defined lattice-theoretically on the lattice of convex ℓ-subgroups?
(Conrad 1978)

No Kenoyer 1982 (to both 9 and 9a)
Czech. Math. J. 1984

9a What about the order-closed convex ℓ-subgroups?
(Redfield '78)

NO to 9A, again by Kenoyer.

9B. Can we recognize the closed convex ℓ-subgroups within $\mathscr{C}(G)$ if we restrict to normal-valued ℓ-groups?

10. Does there exist a finitely presented ℓ-group with insoluble word problem?

(AMW Glass 1972)

TAMS (1983)

Yes: Glass & Gurevich 13·VI·80 . 1280

Given a Turing (Truth Table) degree $\delta < 0'$, does there exist a finitely presented ℓ-group whose word problem has degree exactly δ? The Glass-Gurevich paper gives $\geq \delta$. It might give $=$.

Note that for many-one or bounded truth table degrees the answer is no (Carl Jockusch,

12. Can every finitely generated recursively presented ℓ-group be ℓ-embedded in a finitely presented ℓ-group? (AMWGlass 1972).

Yes: Glass 2006 J. Algebra 301 509-530.

If the original ℓ-group is abelian, then the positive answer had been previously obtained by Glass & Marra (J. London Math. Soc. 68 (2003), 545-562.

The fully analogue of the Boone-Higman Theorem has been obtained for ℓ-groups by Glass J. Group Theory 2008 (appearing)

2.viii.07

12. Characterize the equationally compact ℓ-groups. JMcNulty Oct78

13 Can one tell from the lattice $C(G)$ of all
19/10/78 convex l- subgroups whether G is
archimedean? Yes if G is normal valued.
Conrad

No Kenoyer 1982
Same example as # 9
This also establishes that $C(G)$
does not determine whether
G is completely distributive

14. Is every l-group order dense in an α-complete l-group?

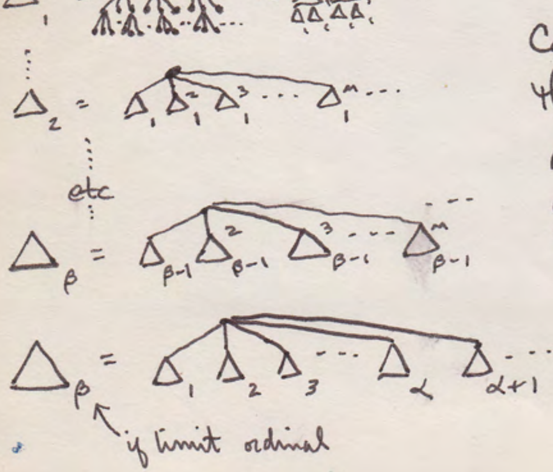
Rick Ball, Oct, 1978

YES, and $G^{i\alpha} = G^{\alpha}\ \forall G$! False. RB-Jan 1982

R.B., April 1980

Does Bernau's (or anybody's) construction of the lateral completion ever require more than 2 steps?

YES! $\Sigma(\Delta_\beta, \mathbb{R})$ requires β-steps



Note that for each root system each root is of finite length. Thus

$$\Sigma(\Delta_\beta, \mathbb{R})^L = V(\Delta_\beta, \mathbb{R})$$

Convince yourself that $\Sigma(\Delta_2, \mathbb{R})$ requires 2 steps and then scream induction!

JRBixler 1/16/82

15. Free products of ordered groups:

A. Is the word problem solvable for free products in the class of abelian ℓ-groups? How about in other classes?

B. Do free products in the class of abelian ℓ-groups have the refinement property? (i.e., does $\sqcup G_i = G = \sqcup H_j \Rightarrow G = \sqcup G_i \cap H_j$?)

C. Let $\mathcal{K}$ be a variety of ℓ-groups and $\mathcal{D}_e$ the variety of distributive lattices with distinguised element e. Let $(G_i \mid i \in \mathcal{I}) \subset \mathcal{K}$. Is ${}^{\mathcal{D}_e}\sqcup G_i$ the sublattice of ${}^{\mathcal{K}}\sqcup G_i$ generated by $\cup G_i$? [Note: true for class of all ℓ-groups (Franchello (1979)) and for class of abelian ℓ-groups (Powell & Tsinakis (1980))]

D. Let $\mathcal{K}$ be a class of ℓ-groups closed under homomorphic images, convex ℓ-subgroups, and free products. Is $\mathcal{K}$ a torsion class?

W. Powell & C. Tsinakis (4/19/80)

A Yes if the Abelian ℓ-groups are finitely presented (by Khisamiev: universal theory of Abelian ℓ-groups is decidable) A.M.W. Glass 12th Feb 88.

2. [illegible] Yes in the class of all ℓ-groups (see 8 [illegible]) Glass, ~~[illegible]~~

16. What o-, l-, po-rings can be embedded in power series rings of the reals? How?

(Redfield, April '80)

17. Is an l-prime l-ring with squares positive a domain?

J. Diem.

Same question with the identity $(x^2)^- = 0$ replaced by $[p(x)]^- = 0$, $p(x)$ an appropriate polynomial.

S. Steinberg

Yes if the ring has a 1 — done by S. Steinberg [note by M. Henriksen]

More generally, if the ring has a nonzero f-element then it is true by J. Ma.

18. Let R be a totally ordered commutative domain with value semigroup Δ. Can R be embedded in the Hahn product $V(\Delta, \mathbb{R})$ (embedded as a domain.)

P. Conrad

NO Example by G. Bergman (~(1974))

19. Find some necessary conditions on a finitary equational theory T so that the T-algebras over Sets and the T-algebras over compact T_2 spaces (0-dim cpt T_2 spaces) are dual.... i.e. how rare are such dualities?

P. Bankston (19 Apr '80)

20. Does there exist a finitely generated simple right orderable group?

(20)' replace simple by perfect (meaning $G = G'$ the derived grp)

A. H. Rhemtulla
April 1980

~~Yes~~ – indeed there exist finitely generated simple subgroups of the group of all piecewise linear members of $A(\mathbb{R})$:

Let $\alpha x_0 = \begin{cases} \alpha & \text{if } \alpha \leq 0 \\ 2\alpha & \text{if } \alpha \geq 0 \end{cases}$ $\qquad \alpha x_1 = \begin{cases} \alpha & \text{if } \alpha \leq 1 \\ \alpha + 2(\alpha - 1) & \text{if } \alpha \geq 1 \end{cases}$

Then the subgroup G of $A(\mathbb{R})$ generated by x_0, x_1 is given by: $\langle x_0, x_1 ; [x_i, z^{2n} x_i^{-1} z^{-2n}] = 1 \quad (n = 1, 2, i = 0, 1)$ $[x_0, z^{2n+1} x_0^{-1} z^{-n}] = 1 \quad (n = 1, 2)$ $[x_1, z^{2m+1} x_0^{-1} z^{-m}] = 1 \quad (m = 0, 1) \rangle$

has G^* sufficiently transitive that Higman's lemma applies to give G' simple – it is generated by Oops!

21 Does every right orderable polycyclic group G possess a series from $\{e\}$ to G with torsion-free abelian factors?

(21)' Replace polycyclic by solvable

(21)'' What if G is an n-Engel group.

A. H. Rhemtulla
April 1980.

(21)'' Yes, by Rhemtulla (Gainesville, 1991)
n-Engel orderable groups ~~[illegible]~~ are nilpotent. (Note: by Martinez)

↓

n-Engel lattice-ordered groups are residually ordered (Medvedev) and any fin. generated n-Engel ordered group is nilpotent (Rhemtulla using results of Zelmanov). So n-Engel lattice-orderable groups have positive answer. Recently (1998) Longobardi & Maj have proved (21)'' when $n \leq 4$. However, at present (16.viii.01), that is the best result for right orderable groups. (AMWZ)

Havas & Vaughan-Lee have proved (2006) that all 4-Engel n generator groups are nilpotent of class $\leq 4n$ (AMWZ).

2.viii.07

22. Is every retractable group
G a unique product group?

A. H. Rhemtulla
April 1980.

23. Does there exist an o-group (non-abelian)

7/9/80 with the property that if $e \ll b \ll a$

then $b \ll b^a$?

with the property that $b \not\approx b^a$?

[If yes then the answer to #8 is no]

TODD FEIL

yes. The free group has such an order

G. Bergman
Summer 1982

A law satisfied by such a group, but not by any of the Medvedev groups M^-, N, M^+ is

$$|[a,b]| \leq |[\,|[a,b]|, |[a,b]|^a]| \text{ for } e \leq b \leq a.$$

Also solved by V.M. Kopytov: ~~Izvest~~ Mat. Sb. 1985, 247-266

24. Is Conrad's representation of the free l-group faithful on some one of the subderived factors? I.e., given a group free ~~ab~~ on a generating set X (having more than one element) does there exist a right ordering of G such ~~that~~ the l-subgroup of A(G) ~~the right regular rep~~ generated by the right regular representation (G, G) is free ~~as~~ an l-group on X?

Steve McCleary
Nov 22, 1980

Yes !! (Kopytov)

25 Arch = category of archimedean ℓ-groups.

$\mathcal{W}$ = cat. of arch. ℓ-groups with weak unit & unit-preserving maps.

What are the epimorphisms in Arch? $\mathcal{W}$?

A. W. Hager & L. Robertson $\approx$ 1976

Solved by Ball & Hager $\approx$ 1984:

for $\mathcal{W}$: $G \leq H$ is epic $\Leftrightarrow$ for each $h \in H^+$, there is countable $E \subseteq H$ s.t.: for any pair $s, t \in Y(H)$ with $\tau(s) = \tau(t)$ and $h(s) \neq h(t)$ there is $k \in E$ s.t. either $k(s) = \infty$ or $k(t) = \infty$. (Here $\tau : Y(H) \to Y(G)$ is the Yosida map induced by the given inclusion of G in H).

26. Can every (distributive) ℓ-semigroup be embedded in a (distributive) ℓ-monoid?

Marlow Anderson

1981

22. Is $\mathbb{Z} \boxplus \mathbb{Z}$ an amalgamation base for the class of all ℓ-groups? $\mathbb{Z} \overleftarrow{\oplus} \mathbb{Z}$? Note that $\mathbb{Z}$ is but $(\mathbb{Z} \boxplus \mathbb{Z}) \overleftarrow{\oplus} \mathbb{Z}$ isn't.

Keith Pierce Jan 16, 1982

28. Do vector lattices form a torsion class? Martinez

Does each special valued ℓ-grp contain a largest convex ℓ-subgroup that is a vector lattice?

Conrad 12/14/91

No, there exists an example of a special-valued ℓ-gp that is not a vector lattice but is the union of convex ℓ-subgps that are vector lattices. The best one can show is that the class of finite-valued vector lattices is a torsion class (due to Conrad - Lin - Nelson)

Nelson 7/27/93 Luminy

29. Are any two scalar multiplications on a vector lattice connected by a l-automorphism?

Note that $V(\Delta, R)$ has this property.

Conrad
1/16/82

30. Does $[x^n, y] = 1_\wedge \;\forall x,y \in G \Rightarrow [x,y] = 1_\wedge \;\forall x,y \in G$ for ℓ-groups G?

J. Smith 1/16/82

For $n = 2$, the answer is yes — assuming Gurchenkov's result concerning covers of $\mathcal{A}$ inside varieties of solvable ℓ-groups is correct.

$y^{-1}xy \le y^{-1}x^n y = x^n \quad \forall x, y, \; x \ge 1$

Hence if $[x^n, y] = 1 \;\forall x, y \in G$, then by Martinez G weakly abelian, so orderable (by Martinez)

By B.H. Neumann, $[x^n, y] = 1 \to [x, y] = 1$ if G orderable.

So Yes. AMW Glass (13/vi/84)

31. Does the free group G of finite rank $n > 1$ have a finite subset S such that there is a unique total order of G making all elements of S positive?

Steve McCleary
Jan 16, 1982

Equivalently, does the free representable l-group R_n have a basic element?
(The free l-group F_n has no basic element, and consequently, if "total order" is changed to "right order", the answer to the above question becomes "NO".)

32. (a) Does any non-abelian variety of ℓ-groups satisfy the amalgamation property?

(b) For what varieties of ℓ-groups is $\mathbb{Z}$ in the amalgamation base? (see 27)

(c) What varieties of ℓ-groups satisfy the divisible embedding property? (see 1)

Powell - Tsinakis
Jan 1982

33.

Which ℓ-groups admit a strictly positive measure function?

$$f(a+b) = f(a) + f(b)$$
$$f(0) = 0$$
$$a > 0 \Rightarrow f(a) > 0$$

B. Bosbach

1. Sept. 82

34. Let G be a (totally) right ordered group and $\mathbb{Q}G = A$ its group algebra over $\mathbb{Q}$ say. It is easy to see that A is an integral domain; can A always be embedded in a field (skew of course)?

12 June '84 Luminy
(but surely much older)

P.M. Cohn

35. June 13, 1984

Let $A = \mathbb{R}[x]$ be lattice-ordered by letting $a(x) = \sum_{i=0}^{n} a_i x^i \geq 0$ if $a_i \geq 0$ for $0 \leq i \leq n$. Is there a lattice-order on $\mathbb{R}(x)$ that extends the order on A? - Note that $\mathbb{R}(x)$ is not a sublattice of the formal Laurent series ordered coefficientwise? M. Henriksen

36. Is there (without the axiom of choice) an example of a lattice-ordered ring A that satisfies

(*) $a \wedge b = 0$ and $c \geq 0 \Rightarrow a \wedge bc = a \wedge cb = 0$

that is not a subdirect sum of totally ordered rings? Note that (*) holds in any zero-ring. [Inspired by the work of Huijsmans and de Pagter] / M. Henriksen

In particular, what if A is a nonzero vector lattice without a basis? 8/10/88

#36 These two definitions are equivalent if and only if the prime ideal theorem for Boolean Algebras. This has been done by D. Feldman & M. HENRIKSEN Indag Math to appear 80

W. Luxemburg

B. Banachewski

TO APPEAR IN ALG. UNIVERSALIS 88

37. Consider $G * \mathbb{Z}$ in the variety of all ℓ-groups.

1) If $G \neq 1$, ? $G * \mathbb{Z}$ has trivial centre

2) If G has soluble word problem, does $G * \mathbb{Z}$?

3) " " " " a transitive representation " " "

4) " " " " doubly transitive " " "

Note: If G cble & $m > 1$, $G * F_m$ has doubly trans rept.
If G uncble & $m \geq |G|$, $G * F_m$ " " " "
(F_k = free ℓ-group on k generators)

AMW Glass
13th June 84

1) Yes AMW Glass (Oct. 1985)

3) & 4) $G * \mathbb{Z}$ has doubly transitive rept on $\mathbb{Q}$ if G is countable (June 1985)

2) Yes AMW Glass (to appear in the Proc. of the Ischia Conference of 2006 in honour of Akbar Rhemtulla) "Free products and Higman-Neumann-Neumann type extensions of lattice-ordered groups".
(also see question 15)

2.viii.07

38 (1) Does the free group on 2 generators have soluble conjugacy problem? (↑ any finite no.)

AMW Glass
13/VI/84.

(2) Do nilpotent class 2 ℓ-groups (which are finitely presented in the class) have soluble conjugacy problem? The answer is yes if one can prove that if $f, g \in G$ (a finitely presented nilpotent class 2 ℓ-group) that are not conjugate in G, then there is a linearly ordered homomorphic image in which the elements aren't conjugate

AMW Glass
24/IV/87

The answer to this part is no (May '90)
It appears in P.S. Moore's Ph.D. thesis
AMWG.

39. Let $w(x, y)$ be an arbitrary word in the free group on $\{x, y\}$. Do there exist $f, g \in A(\mathbb{R})$ (= the group of order-preserving permutations of the real line) such that $w(f, g)$ fixes no point?

If so, then for any $p \in A(\mathbb{R})$ and any $w(x_1, x_2, \dots, x_n)$ in a free group the equation $w(x_1, \dots, x_n) = p$ has a solution.

What about $A(\Omega)$ (doubly transitive)?

C. Holland
13 June 1984
Luminy.

"Yes" to both. Samson Adeleke
Fall, 1986

see forthcoming paper of Adeleke + Holland

40. Let $(\mathfrak{N}_k)_l$ - the variety of nilpotent l-group of class $\le k$, $\mathfrak{H}_l$ - the variety of l-group with identity $x^{-1}|y|x \le |y|^2$. Is this equality $\mathfrak{H}_l = \bigvee_{k=1}^{\infty} (\mathfrak{N}_k)_l$ true? ~~[illegible]~~ By "$\bigvee$" - we mean a joint in the lattice of all varieties of l-groups.

V.M. Kopytov
15.06.1984
Luminy

No. Vasily Bludov and AMWGlass, TAMS 385 (2006), 5179-5192.

2.VIII.07

41. Let F_n be the free lattice-ordered group on n generators, n finite.

(a) Can $F_m \equiv F_n$ if $m \neq n$?

(b) Same problem for free abelian lattice-ordered groups.

($F_2 \not\equiv F_3$ in the Abelian case)

AMW Glass
Luminy 1984

(a) No — (Beynon, Glass, Macintyre & Point (July 84))
$Th(F_2)$ is decidable. We suspect that $Th(F_n)$ is undecidable if $n > 2$.

(b) No & $Th(F_n)$ is undecidable iff $n > 2$

"Free abelian lattice-ordered groups" Annals of Pure & Applied Logic 134 (2005), 265-283.

42. (b) Can a normal-valued ℓ-gp be recognized from the underlying lattice?

(MR Darnel 9/84)

(a) ~~[illegible]~~ Does there exist any proper ℓ-variety whose elements are recognizable from the underlying lattice?

No to both, Hungarian Example Czech Math J. 1984

43. Does every ℓ-group with a unique addition have to be archimedean?

(M.R. Darnel 9/84)

Yes, Jakubik 1990.

MRD

Czech. Math. J. 41 (116) (1991), 559 - 563

44. $G = \{ f \in A(\mathbb{R}) \mid xf + 1 = (x+1)f, \ \forall x \in \mathbb{R} \}$ is not divisible. Is any periodic o-permutation group divisible?

Spencer Hurd, May 1985

45. Suppose the free product of vector lattices $A \amalg B$ is (isomorphic to) a free vector lattices. Is A necessarily free?

J. Madden 1986

The analogous problem for boolean algebras is solved as follows:

If $A \amalg B$ is free and <u>finitely generated</u>, then the stone space of $A \amalg B$ is 2^n (discrete space with 2^n elements). So, stone spaces for A and B must be (respectively) 2^{n_1} & 2^{n_2} $(n_1 + n_2 = n)$. Thus, A & B are free. <u>However</u>: If A is the boolean algebra of subsets of 3 & B is the free boolean algebra on ω generators, then $A \amalg B$ is free.

Returning to the vector-lattice problem, Madden showed in his thesis that <u>if A & B are finitely generated</u>, then the answer to 45 is "Yes". But the example with boolean algebras suggests that without the finiteness assump the answer might be "no".

to 45

J.M. (following discussion with Jipsen. 2004 conference in Gainesville

46. If G_0 is a (finitely generated) ℓ-subgroup of G, and H_0 is of H, is the ℓ-subgroup of $G * H$ (the free product of G & H in the class of all ℓ-groups) generated by $G_0 \cup H_0$, the free product of G_0 & H_0?

AMW Glass
(10/1/86)

47.

(short := does not countain an uncountable well-ordered subset)

Does there exist a short ordered commutative domain, whose quotient field, with the canonically extended order, is not short ?

Claus Borregaard

Yes, there exists a short commutative domain whose quotient field contains an increasing sequence of length α for any ordinal $\alpha < \mathfrak{c}^+$.

On the other hand if a commutative domain D is short than the power of it's quotient field F is

$|F| = |D| \leq \mathfrak{c}$.

Krzysztof Ciesielski
February 1986

48. Let G be a t.o. nilpotent group, and G^d the divisible hull of G. Do G & G^d satisfy the same equations*?

In other words, can you invalidate a law by adding roots? J. Madden, '88

* These were supposed to be <u>l-group</u> equations in the original question, but I don't even know the answer for just group equations.

Aug 1988

For group identities - <u>yes</u> - old work of G. Baumslag. ≈ 1960

For l-group identities - no - S. Gurchenkov (1988).

N. Medvedev 1991

Answer: No; see paper by Gurchenkov (1988)

MRD

49. Let G be a totally ordered group, and let $w = w(x, y, \ldots)$ be a word in a free group.

Suppose that for some substitution $\bar{x}, \bar{y}, \ldots \in G$, we have $w(\bar{x}, \bar{y}, \ldots) > e \in G$.

Does it follow that also for some substitution $\bar{\bar{x}}, \bar{\bar{y}}, \ldots \in G$, we have $w(\bar{\bar{x}}, \bar{\bar{y}}, \ldots) < e$?

C. Holland, Jan. 9, 1988.

No. Let G be the nilpotent group generated by two elements a, b and defined in the variety of nilpotent groups of the nilpotent class ≤ 6 by the following relations:

$$[[[[[b,a]a]a]a]a] = [[[[[b,a]a]a]a]b] = [[[[[b,a]a]a]b]b] =$$
$$= [[[[[b,a]a]b]b]b] = [[[[[b,a]b]b]b]b] = [[[[ba]a]b],[ba]] =$$
$$= [[[[ba]a]a],[b,a]] = [[[b,a]b],[[b,a]a]] = e$$

It is „evident" that G is without torsion and nilpotent of class 6. From calculations follows that all nontrivial values of group word $w(x,y) = [[[[yx]y]y],[y,x]]$ in the group G have the same sign. This fact gives the negative solution of problem 49.

HMegbegel 12.12.1991

50. Let K be a field. Then

$P(K) = \min_n \left(\sum K^2 = \sum^n K^2\right)$ is called the Pythagoras Number of K. It is known that every number 2^m and 2^m+1 occurs as Pyth. Number of some (formally real) field. Can P take any other value?

A. Prestel, 88

51. Let a group G be a subdirect product of finite p-groups and a subdirect product of finite q-groups where p and q are different prime numbers. Then G is right orderable. Prove.

N. Ya. Medvedev 11.1991.
(Bowling Green)

52. For an arbitrary natural number k construct a group G_k such that:

a) G_k is not orderable;

b) every k-generated subgroup of G_k is orderable.

N. Ya. Medvedev 11. 1991.
(Bowling Green).

53. Let M^+, M^-, M_0 be the soluble representable covers of l-variety of abelian l-groups $\mathfrak{A}_e$ (M. Anderson, T. Feil, Lattice-ordered groups, p. 60) and K, B be nonsoluble representable covers of $\mathfrak{A}_e$ (Kopytov V. M. Mat. Sb., 1985, 126(168), p. 247-266.; Bergman G. Comm. in algebra, 1984, 12(19), p. 2315-2333) in the lattice of l-varieties L.

For each of these l-varieties M^+, M^-, M_0, K, B find a finite basis of identities or prove that there is none.

N. Ya. Medvedev. 11. 1991.
(Bowling Green)

54. Does there exist (if so, how many) any partial order $\leq$ on $\mathbb{Q}$, which is not a total order and such that $(\mathbb{Q}, \leq)$ is narrow (ie. $\mathbb{Q}$ does not contain any infinite subset which is trivially ordered). I only consider orders compatible with addition.

Have fun and let me know what you found

Paulo Ribenboim

55. $\mathbb{R}$ = chain of real numbers,
$\mathbb{Q}$ = chain of rational numbers

Thm (Darnel, Giraudet, McCleary).
Composition and its reverse are the only group operations (with the usual identity) making the lattice $A(\mathbb{R})$ an l-group.

This "should" be true also of $A(\mathbb{Q})$, but:

Is it?

Steve McCleary

56. Let $G = \mathbb{Z} \overleftarrow{wr} \mathbb{Z}$, $M = \overleftarrow{\Sigma}(\mathbb{Z}, \mathbb{Z})$ in G (the maximal convex subgroup of G).

~~Define $D_n(G)$ to~~ Let $y \in G^+ \ni$ My generates G/M. Define $D_n(G)$ to be a lex extension of $\sum_{i=1}^{n} M$ by $\langle t \rangle$, where $t^{-1}(m_1, \dots, m_n)t = (y^{-1}m_n y\, m_1, m_2, \dots, m_{n-1})$.

Let $\mathcal{D}_n(G)$ be the ℓ-variety ~~of~~ gen by $D_n(G)$. $\forall$ prime p, does $\mathcal{D}_p(G)$ cover ℓ-over(G)?

Darnel

57 Is each abelian a~~*~~ closed l-gp a vector lattice?

Conrad 12/14/91

58. Does every ℓ-group that is a subdirect product of integers have a unique addition? (Conrad-Darnel, 1991)

59. Can the complex numbers be lattice-ordered as a ring.

Nelson Luminy 93

60. Can ~~the~~ every semilinear order of a group be extend to right order?

Kopytov Luminy '93

61. An automorphism φ of an l-group G is called order-reversing if $x \leq y$ implies $x\varphi \geq y\varphi$ for all $x, y \in G$.

Let F_r be a free non-abelian l-group of rank r $(r \geq 2)$. Does there exist a non-trivial order-reversing automorphism φ of F_r?

N. Ya. MEDVEDEV Luminy 93.

69. Let I and J be totally ordered sets, and denote by $\mathbb{R}^I$ and $\mathbb{R}^J$ the lexicographically ordered sets of sequences of reals, with well ordered support in I (respect. in J).

Assume $\mathbb{R}^I \simeq \mathbb{R}^J$ as ordered sets.
Does it imply $I \simeq J$ as ordered sets?
(Note: if I and J are ordinals, y can prove that.)

Salma Kuhlmann 29.07.93
University of Heidelberg

NO. For $\mathbb{Z}$ = integers and $\mathbb{N}$ = natural numbers, $\mathbb{R}^{\mathbb{Z}} \simeq \mathbb{R}^{\mathbb{N}}$.
Proof: Each of the two sets has the form $\sum_{n\in\mathbb{Z}} K_n$ where $K_n \simeq \mathbb{R}^{\mathbb{N}}$, as follows:

(i) for $\mathbb{R}^{\mathbb{N}}$, $K_n = \begin{cases} \{(\tfrac{n}{2}, \dots)\} & \text{if } n \text{ is even} \\ \{(x, \dots) \mid \tfrac{n-1}{2} < x < \tfrac{n+1}{2}\} & \text{if } n \text{ is odd} \end{cases}$

(ii) for $\mathbb{R}^{\mathbb{Z}}$, $K_n = \begin{cases} \{(\dots, \underset{-1}{0}, \underset{0}{0}, \underset{1}{x}, \dots)\} & \text{if } n = 0 \\ \{(\dots, 0, 0, \underset{n}{x}, \dots) \mid 0 < x\} & \text{if } n > 0 \\ \{(\dots, 0, \underset{n}{x}, \dots) \mid x < 0\} & \text{if } n < 0 \end{cases}$

Charles Holland
Bowling Green, Fall '93

63. Let G be an O-gp with no free (non-abelian) subsemigp. Is G locally nilpotent?

Akbar Rhemtulla (1993)

64, Henriksen & Isbell ("Lattice-ordered rings and function rings," Pac. J. Math 12(1962), 533-65) define* a totally ordered commutative ring with 1 (possibly having nilpotent elements) to be <u>formally real</u> if it is isomorphic to a quotient of a total ordering of $\mathbb{Z}[\bar{x}]$ by a convex ideal, I.e., A is formally real if there is a total order τ on $\mathbb{Z}[\bar{x}]$ and a τ-convex ideal I s.t. $A \cong (\mathbb{Z}[\bar{x}], \tau)/I$.† <u>Conjecture</u>. Every ring with two generators is formally real.

J. Madden Feb 1998

* see Proposition 4.7, page 547.

† This is an <u>order</u>-isomorphism. $\bar{x}$ is the set of generators; $\bar{x} = (x_1, x_2, \dots)$. $\mathbb{Z}[\bar{x}]$ is the ring of polynomials with integer coefficients.

<u>Note</u>: A weaker — therefore more likely — conjecture is that every 2-generator $\mathbb{R}$-algebra is formally real. This might be a lot easier due to <u>J. Erdös'</u> (not P.) theorem.

65. (The Kohls problem)

X any completely regular space

~~P a prime ideal of C(X)~~. $p \in X$.

If $C(X)/O_p$ is totally ordered does it follow that $C(X)/O_p$ is a valuation domain? ($O_p = \{ f \in C(X) ; f$ vanishes on a nbhd of $p \}$)

(This problem was first posed by C.W. Kohls in the late fifties.)

J. Martinez,
March 1998.

66. For an arbitrary chain T, is the wreath product $\mathcal{W}r_T(\mathbb{R}, \mathbb{R})$ of reals a-closed? (This is true if T is an ordinal.)

M. Darnel 1998

67. Is every L^*-ring O^*? Same question for groups. (L^*-ring is a ring every p.o. of which extends to a lattice order). (O^* - every p.o. extends to a total order).
[Piotr Wojciechowski, Gainesville 98]

Partial answer known: Every F^*-ring is O^*.
(Jingjing Ma and Piotr Wojciechowski "F^*-rings are O^*"; Order 2000)

68. Classify the hyper-archimedean ~~ideals~~ l-kernels (ring ideal, convex l-subgp). of a semiprime f-ring A.

Furthermore, what conditions ~~of~~ on A characterize when a h.a. kernel is a finite intersection of real maximal ideals.

Note: for $C(X)$, this last statement is true.

for $C(X, \mathbb{Z})$, X cpt. every l-ideal is a hyper-arch. kernel.

[Warren McGovern
Gainesville, 98]

~~A~~ 9. Is every Prufer singular f-ring (commutative w/ 1) Bezout.

Known: A, a singular f-ring, is Prufer $\Leftrightarrow A/P$ is Prufer
$P \in Min(A)$.

A is Bezout $\Leftrightarrow A/P$ is Bezout
$P \in Min(A)$.

So restated, $\exists$? a totally ordered Prufer domain w/ least element 1 which is not Bezout?

Posed by [Finn, Martinez, McGovern].

10. If $\bar{H}$ denotes the normal closure of a subgroup H of a group G, then is

$$\overline{H \cap K} = \bar{H} \cap \bar{K}$$

equivalent with the claim that G is Hamiltonian?
(It is true if G is finite).

B. Šešelja (1998)
in Nanjing

71. Does the class of right ordered groups fail the Amalgamation Property?
Ditto Conrad right ordered?
(Note the classes of right orderable groups, normal-valued l-gps & all l-groups, all fail the Amalg. Prop — Kopytov & Medvedev
K.R.Pierce
K.R. Pierce)

AMWGlass 8.vii.98
Nanjing

A result of Bludov gives the failure of the amalgamation property of right ordered groups. It uses the infinite dihedral group and a thm of George Bergman (see, a version in English in Ch. 11 of "Partially ordered groups" by AMWG).

72. If G is a finitely generated orderable Abelian-by-nilpotent group and every order on G is weakly Abelian, then must G be nilpotent?

AMW Glass, Dresden
(+ AH Rhemtulla) 16.viii.01

If yes, then every finitely generated orderable soluble-by-finite group in which every order is weakly Abelian must also be nilpotent.

(AMWG 16.viii.01)

73. X is called a <u>quasi P-space</u> if every prime Z-ideal is minimal or maximal. Every known example of one is (strongly) zero dimensional. Is there an infinite (normal) connected quasi P-space?

*Must a free union of quasi P-spaces be quasi P? (All spaces considered are Tychonoff)

March 8, 2002

M. HENRIKSEN

* Yes, to this question. Proof (unfortunately) requires interpreting "quasi P" in terms of Z-dimension. The Z-dimension of $C(X)$, $\dim_Z(C(X))$ is the length k of the longest chain $p_0 < p_1 < \ldots < p_k$ of prime Z-ideals. (J. Martinez, March, '03)

74. Let G be an archimedean ℓ-group.

A Johnson-representation of G is an isomorphism $G \approx \hat{G} \subseteq D(Y)$ s.t.

Y is locally compact,

$\forall p \notin$ closed F $\exists g$ s.t.

$0 < \hat{g}(p) < +\infty$ and $\hat{g} \mid F = 0$,

$\forall g$, $\beta\hat{g}(\beta Y - Y) \subseteq \{0, \pm\infty\}$.

<u>Question.</u> Does <u>every</u> G have a J-representation?

n.b. D.G. Johnson, Proc. London Math. Soc 1962, shows

G also a semi-prime f-ring (<u>no</u> <u>identity</u> <u>assumed</u>) $\Rightarrow$ Yes, even as an f-ring.

A. Hager 2/8/03.

75 Does there exist a Tychonov space X, (not pseudocompact) such that the set $\beta X \setminus X$ is infinite, but for every $f \in C(X)$ the set $(f)_\infty :=$
$= \{p \in \beta X \mid f$ does not extend continuously to $X \cup \{p\}\}$ is finite?
If so, characterize these spaces X in other ways.

M. Knebusch, March 5, 2004
(Fakultät f. Mathematik
Universität D-93040 Regensburg)

76. Why are almost all l-group varieties defined by identities with 2 variables?
(M. Ploščica, Dresden 2004)

77. (a) If G_1, G_2 o-groups sharing a common subgroup S, does there exist an ℓ-group L and ℓ-embeddings $\tau_j : G_j \to L$ agreeing on S $(j = 1, 2)$?

<u>Note</u>: $G_1 \underset{S}{*} G_2$ is right orderable and so embeddable in a lattice-<u>orderable</u> group (Bludov & Glass) but the question asks for ℓ-embeddings.

(b). The infinite Dihedral group can be expressed as $\langle a \rangle \underset{a^2=b^2}{*} \langle b \rangle$ and is right orderable but not lattice-orderable. If G_1, G_2 are o-groups with a common subgroup S and $G_1 \underset{S}{*} G_2$ is lattice-orderable, must $G_1 *_S G_2$ be orderable?

AMWGlass
Aug '07.

78. Is there a finitely presented orderable group with insoluble word problem?

78': Weaker Question: Is there a finitely presented lattice-orderable group with insoluble word problem?

[Note: There is a finitely presented right orderable group with insoluble word problem (Bludov, Giraudet, Glass & Sabbagh)]

Arrwgian
Aug '07.