Quasivarieties of commutative residuated lattices

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Therefore, if K is a class of algebras, then $\mathsf{Q}(\mathsf{K}) = \boldsymbol{ISPP_u}(\mathsf{K})$ is the quasivariety generated by K.

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If Q is a quasivariety and $\mathbf{A} \in \mathsf{Q}$, a **relative congruence** of $\mathbf{A}$ is a congruence θ such that $\mathbf{A}/\theta \in \mathsf{Q}$; relative congruences form an algebraic lattice $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ and for any congruence lattice property P we say that $\mathbf{A} \in \mathsf{Q}$ is **relatively** P if $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ satisfies P.

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So for instance $\mathbf{A}$ is **relatively subdirectly irreducible** if $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ has a unique minimal element; since clearly $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ is a meet subsemilattice of Con($\mathbf{A}$), any subdirectly irreducible algebra is relatively subdirectly irreducible for any quasivariety to which it belongs.

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For a quasivariety Q we denote by Q_{RSI} the class of relatively subdirectly irreducible algebras in Q

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Theorem

Let Q *be any quasivariety.*

1. *(Mal'cev) Every* $\mathbf{A} \in \mathsf{Q}$ *is a subdirect product of algebras in* Q_{RSI}*.*
2. *(Czelakowski and Dziobiak) If* $\mathsf{Q} = \mathsf{Q}(\mathsf{K})$*, then* $\mathsf{Q}_{RSI} \subseteq \boldsymbol{ISP}_u(\mathsf{K})$*.*

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Lattices of subquasivarieties are in general very complex. A quasivariety Q is **Q-universal** (Sapir) if for any other quasivariety Q' of finite type, $\Lambda_q(\mathsf{Q}')$ is a homomorphic image of a sublattice of $\Lambda_q(\mathsf{Q})$.

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Lemma

For every Q-universal quasivariety Q

- *the free lattice on* ω *generators is embeddable in* $\Lambda_q(\mathsf{Q})$*;*
- $|\Lambda_q(\mathsf{Q})| = 2^{\aleph_0}$.

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Lemma

(Adams-Dzobiak) Let Q *be a quasivariety such that* $\mathbf{V}(\mathsf{Q})$ *is congruence distributive and has the congruence extension property. If* Q *contains an infinite family of simple algebras, such that none is embeddable in any other, then* Q *is Q-universal.*

The low hanging fruits

A quasivariety Q is locally finite if every finitely generated algebra in Q is finite, and it is finitely generated if it is generated by finitely many finite algebras.

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The following facts are easy to check:

- Q is locally finite or finitely generated if and only if $\boldsymbol{V}(\mathsf{Q})$ is such;
- for any quasivariety Q and any subquasivariety Q′ of Q, $\boldsymbol{V}(\mathsf{Q}')$ is the smallest variety V such that $\mathsf{Q}' \subseteq \mathsf{V} \cap \mathsf{Q}$.

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An algebra $\mathbf{A}$ is **weakly projective** in a class K of algebras (of the same type) if for all $\mathbf{B} \in \mathsf{K}$, if $\mathbf{A} \in \boldsymbol{H}(\mathbf{B})$ then $\mathbf{A}$ is embeddable in $\mathbf{B}$.

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We denote by $[\mathsf{K} : \mathbf{A}] = \{\mathbf{B} \in \mathsf{K} : \mathbf{A} \text{ is not embeddable in } \mathbf{B}\}$.

A test for primitivity

Theorem

(Gorbunov1998) Let Q *be a locally finite quasivariety of finite type. Then the following are equivalent:*

1. Q *is primitive;*
2. *if* **A** *is a finite relative subdirectly irreducible algebra in* Q, *then* [Q : **A**] *is a quasivariety that is equational relative to* Q*;*
3. *every finitely generated relative subdirectly irreducible algebra* **A** $\in$ Q *is weakly projective in* Q*;*
4. *every finite relative subdirectly irreducible algebra* **A** $\in$ Q *is weakly projective in the class of finite members of* Q.

Note that condition 3. is sufficient for primitivity even if Q is not locally finite.

Basic hoops and Wajsberg hoops, a case study

A **commutative residuated lattice** is an algebra $\mathbf{A} = \langle A, \vee, \wedge, \cdot, \rightarrow, 1\rangle$ where

1. $\langle A, \vee, \wedge\rangle$ is a lattice;
2. $\langle A, \cdot, 1\rangle$ is a commutative monoid;
3. $\rightarrow$ and $\cdot$ form a residuated pair, i.e. $x \cdot y \leq z$ iff $x \leq y \rightarrow z$.

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A **Wajsberg hoop** is a basic hoop that satisfies Tanaka's equation

$$(x \rightarrow y) \rightarrow y \approx (y \rightarrow x) \rightarrow x.$$

The Mundici's construction

Let **G** be a lattice ordered abelian group; if u is a strong unit of **G** we can construct a bounded Wajsberg hoop $\Gamma(\mathbf{G}, u) = \langle [0, u], \rightarrow, \cdot, 0, u \rangle$ where $ab = \max\{a + b - u, 0\}$ and $a \rightarrow b = \max\{u - a + b, u\}$.

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We now define some useful Wajsberg chains:

- the finite Wajsberg chain with $n + 1$ elements $\mathbf{Ł}_n = \Gamma(\mathbb{Z}, n)$;
- the infinite Wajsberg chain $\mathbf{Ł}_n^\infty = \Gamma(\mathbb{Z} \times \mathbb{Z}, (n, 0))$;
- the infinite Wajsberg chain $\mathbf{Ł}_{n,k} = \Gamma(\mathbb{Z} \times \mathbb{Z}, (n, k))$;
- the infinite bounded Wajsberg chain $[0, 1]_{\mathbf{Ł}} = \Gamma(\mathrm{R}, 1)$, i.e. the real interval with operations induced by the *Wajsberg norm*. i.e $xy = \max(x + y - 1, 0)$, $x \to y = \min(1 + x - y, 1)$;
- the infinite bounded Wajsberg chain $\mathbf{Q} = \Gamma(\mathbb{Q}, 1) = \mathbb{Q} \cap [0, 1]_{\mathbf{L}}$;
- the unbounded Wajsberg chain $\mathbf{C}_\omega$ that has as universe the free group on one generator, where the product is the group product and $a^l \to a^m = a^{\max(l-m, \mathbf{0})}$;
- finally we fix once and for all an irrational number $\alpha \in [0, 1]$ and we let X be the totally ordered dense subgroup of $\mathbb{R}$ generated by α and 1; then $\mathbf{S}_n = \Gamma(X, n)$.

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It is obvious that the family $\{\mathbf{Ł}_{p-1} : p \text{ prime}\}$ satisfies the criterion, so the variety WH of Wajsberg hoops (and a fortiori every variety of commutative residuated lattices containing it, including basic hoops) is Q-universal.

On the other hand there are many primitive quasivarieties of basic hoops, due to the following:

Lemma

(Spinks-Veroff) Let $\mathbf{A}$ *be a basic hoop and let* e *be an idempotent of* $\mathbf{A}$*; then for any* $a, b \in A$ *and any idempotent* e *we have*

$$\begin{aligned} e \to (a \to b) &= (e \to a) \to (e \to b) \\ e \to ab &= (e \to a)(e \to b). \end{aligned}$$

So then the map $g_e : a \longmapsto e \to a$ *is an endomorphism of* $\mathbf{A}$*.*

Lemma

Any finite basic hoop is weakly projective in the class of finite basic hoops.

Proof.

Let $\mathbf{A}$ be any finite basic hoop and suppose that $\mathbf{A} \in \boldsymbol{H}(\mathbf{B})$ for some finite $\mathbf{B} \in \mathsf{BH}$. Then there is a filter F on $\mathbf{B}$ such that $\mathbf{B}/\theta_F \cong \mathbf{B}$; since $\mathbf{B}$ is finite such filter must have a minimum e which is necessarily idempotent and moreover $F = [e)$. Since $g_e(b) = 1$ if and only if $e \leq b$, we have that

$$\mathbf{A} \cong \mathbf{B}/\theta_F = g_e(\mathbf{B}) \subseteq \mathbf{B}.$$

Hence $\mathbf{A} \in \boldsymbol{S}(\mathbf{B})$ and $\mathbf{A}$ is weakly projective in the class of finite basic hoops. □

Theorem

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For instance Čitkin used these techniques to classify all primitive subvarieties of Heyting algebras...

Quasivarieties generated by chains

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However for these quasivarieties we have a tool: it can be shown that if K, K' are classes of commutative and integral chains, then $\mathsf{Q}(\mathsf{K}) = \mathsf{Q}(\mathsf{K}')$ if and only if $\boldsymbol{ISP_u}(\mathsf{K}) = \boldsymbol{ISP_u}(\mathsf{K}')$.

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This allows us to describe all the quasivarieties of Wajsberg hoops generated by chains.

A **presentation** is a set $\{I, J, \{J_r : r \in J\}, L, K\}$ of subsets of $\mathbb{N}$ with the following properties:

- for any $r \in J$, $J_r \subseteq \{j : j \mid r\}$;
- $K \subseteq \{0\}$;
- if $K = \{0\}$, then $J = L = \emptyset$.

To any presentation we associate sets of Wajsberg chains in the following way: where

$$Q_I = \{\mathbf{Ł}_i : i \in I\}$$

$$Q_J = \bigcup_{r \in J} \{\mathbf{Ł}_{r,j} : j \in J_r\}$$

$$Q_L = \{\mathbf{S}_l : l \in L\}$$

$$Q_K = \{\mathbf{C}_\omega\} \text{ if } K = \{0\} \text{ and } \emptyset \text{ if } K = \emptyset.$$

Theorem

Let Q *be quasivariety of Wajsberg hoops generated by chains; then there is a presentation* $I, J, \{J_r : r \in J\}, L, K \subseteq \mathbb{N}$ *such that*

$$\mathsf{Q} = \mathsf{Q}(Q_I \cup Q_J \cup Q_L \cup Q_K).$$

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The proof is a generalization of an older proof by Gispert for universal classes of MV-algebras.

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The proof is a generalization of an older proof by Gispert for universal classes of MV-algebras.

We can also describe the inclusion relation between quasivarieties generated by chains (very technical!).

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Lemma

(Agliano-Montagna 2003) Let K *be any class of basic hoops; then* $\mathsf{Q} = \mathsf{Q}(\mathsf{K})$ *is a variety if and only if all finitely generated totally ordered members of* $\boldsymbol{V}(\mathsf{K})$ *are in* $\boldsymbol{ISP}_u(\mathsf{K})$*. If* Q *has the FEP, then* $\mathsf{Q} = \mathsf{Q}(\mathsf{K}_{fin})$*.*

Theorem

Let $\mathbf{A}_1, \dots, \mathbf{A}_n$ be totally ordered Wajsberg hoops; if for $i = 1, \dots, n$

- *$\mathbf{Ł}_n$ is embeddable in $\mathbf{A}_i$ for all n, or*
- *$\mathbf{A}_i$ is finite, or*
- *$\mathbf{A}_i$ is cancellative, or*
- *$\mathbf{A}_i$ is infinite, bounded and the rank of $\mathbf{A}$ is equal to $d_{\mathbf{A}}$,*

then $\mathrm{Q}(\mathbf{A}_1, \dots, \mathbf{A}_n)$ is a variety.

The previous result can be easily extended to basic hoops that are finite **ordinal sums** of Wajsberg hoops.

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Now a finite chain is necessarily bounded and in WH is also 1-generated. If it were a subalgebra of $\mathbf{F}_{\mathsf{WH}}(1)$, then the image of the bound would be idempotent in the free algebra. But it is clear from its description (Aglianò-Panti 1998) that the only idempotent therein is 1.

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The same argument can be used for finite chains in BH; in this case we have to tweak the description of the free n-generated BL-algebras, i.e. the bounded version of basic hoops, (Aguzzoli and Bova) and adapt it to the case at hand, but the conclusion is the same.

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We observe that $\mathsf{Q}(\mathbf{C}_\omega) = \mathsf{C}$ and $\mathsf{Q}(\mathbf{Ł}_1) = \boldsymbol{V}(\mathbf{Ł}_1)$ are clearly atoms in $\Lambda_q(\mathsf{WH})$ and are the only atoms generated by chains.

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Since $\mathbf{Ł}_1 \in \mathsf{Q}$ for any quasivariety generated by a bounded Wajsberg hoop, $\mathbf{A}$ must be unbounded and contained in $[\mathsf{WH} : \mathbf{Ł}_1]$.

THANK YOU!