Sublocale lattices and the T_D-duality

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Background

Let L be a frame.

- A $p \neq 1$ is called **covered prime** if $\bigwedge_i x_i = p \implies x_i = p$ for some i.

Example

If $L = \Omega(X)$, an $X - \overline{\{x\}}$ is covered iff $\{x\}$ is locally closed.

- Thus, a T_0-space X is T_D iff every $X - \overline{\{x\}}$ is a covered prime.

Denote

$$\mathrm{pt}_D(L) := \{p \in L \mid p \text{ is a covered prime.}\}$$

Background

- A locale L is **T_D-spatial** if $L \cong \Omega(X)$ for a T_D-space X.
- Equivalently, a locale is T_D-spatial if and only if every element is a meet of covered primes.

▶ B. Banaschewski and A. Pultr *Pointfree aspects of the T_D axiom of classical topology*, Quaestiones Mathematicae **33** 2010.

- A **D-homomorphism** between frames is a frame homomorphism with the additional property that its right adjoint preserves covered primes.
- Let Frm_D be the category of frames and D-homomorphisms between them. There is an adjunction

$$T_D\text{-Top} \underset{\Sigma'}{\overset{\Omega}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Frm}_D^{op}$$

D-sublocales

If p is a prime in a sublocale S, then it is a prime in L.

However, arbitrary sublocales are not well-behaved w.r.t. the property of being a covered prime: we may have p which are covered in some $S \subseteq L$ but not in L.

In fact, every prime is covered in the sublocale $\mathfrak{b}(p) = \{1, p\}$!

In

- I. Arrieta, A.L. Suarez, *The coframe of D-sublocales and the T_D-duality*, Topology and its Applications **291** 2021.

we studied sublocales that satisfy that extra property.

Definition

A sublocale S of L is a *D-sublocale* if $\mathrm{pt}_D(S) \subseteq \mathrm{pt}_D(L)$ — i.e. iff $\mathrm{pt}_D(S) = S \cap \mathrm{pt}_D(L)$ (iff the associated surjection $L \twoheadrightarrow S$ is a D-homomorphism).

The system of D-sublocales

Now, we define

$$\mathcal{S}_D(L) = \{S \subseteq L \mid S \text{ is a } D\text{-sublocale of } L\} \subseteq \mathcal{S}(L).$$

$\mathcal{S}_D(L)$ is not closed under binary intersections. What can be said about this system?

Theorem

$\mathcal{S}_D(L)$ is a dense[1] subcolocale of $\mathcal{S}(L)$. In particular, it is a co-frame.

Let now $\mathcal{S}_b(L)$ be the Booleanization of $\mathcal{S}(L)$. Then

Corollary

For any locale L, one has $\mathcal{S}_b(L) \subseteq \mathcal{S}_D(L)$.

[1]Shorthand for "$\mathcal{S}_D(L)^{op}$ is a dense sublocale of $\mathcal{S}(L)^{op}$"

Some examples of D-sublocales:

1. All joins of complemented sublocales are D-sublocales. In particular, open, closed, complemented, locally closed sublocales are D-sublocales.
2. Every pointless sublocale is a D-sublocale.

The definition of $\mathcal{S}_D(L)$ allows us to have a well-defined monotone map $\mathrm{pt}_D \colon \mathcal{S}_D(L) \longrightarrow \mathcal{P}(\mathrm{pt}_D(L))$. On the other hand, consider the map $\mathfrak{M} \colon \mathcal{P}(\mathrm{pt}_D(L)) \longrightarrow \mathcal{S}_D(L)$ given by $\mathfrak{M}(Y) = \bigvee_{p \in Y} \mathfrak{b}(p)$.

Proposition

There is an adjunction

$$\mathcal{P}(\mathrm{pt}_D(L)) \underset{\mathrm{pt}_D}{\overset{\mathfrak{M}}{\underset{\bot}{\rightleftarrows}}} \mathcal{S}_D(L)$$

Moreover

- $\mathfrak{M}(\mathrm{pt}_D(S))$ *is the* T_D*-spatialization of* S*—i.e. the largest* T_D*-spatial* D*-sublocale of* S*.*
- *The fixpoints of* $\mathfrak{M} \circ \mathrm{pt}_D$ *are the* T_D*-spatial* D*-sublocales,*
- $\mathrm{pt}_D \circ \mathfrak{M}$ *is always the identity,*

Therefore, we will write

$$\mathrm{sp}_D^L = \mathfrak{m} \circ \mathrm{pt}_D \colon \mathcal{S}_D(L) \to \mathcal{S}_D(L)$$

and we shall refer to it as the **pointwise T_D–spatialization** operator, as it sends every D-sublocale to its T_D-spatialization.

Lemma

For a frame L, the map $sp_D \colon \mathcal{S}_D(L) \to \mathcal{S}_D(L)$ is an interior operator which preserves joins.

Hence, the map $sp_D^L \colon \mathcal{S}_D(L) \twoheadrightarrow sp_D[\mathcal{S}_D(L)]$ is a coframe surjection whose codomain is the ordered collection of the T_D-spatial D-sublocales of L.

However, $\mathcal{S}_D(L)^{op}$ is a sublocale of $\mathcal{S}(L)^{op}$, and hence in particular a frame in its own right. Accordingly, we can also compute the T_D-spatialization of $\mathcal{S}_D(L)^{op}$, the **global T_D-spatialization**.

Proposition (Sanity check)

We have $sp_D^L[\mathcal{S}_D(L)]^{op} = sp_D^{\mathcal{S}_D(L)^{op}}(\mathcal{S}_D(L)^{op})$—i.e. the local and the global T_D-spatialization coincide.

Total spatiality

Recall that L is said to be **totally spatial** if every sublocale of L is spatial.

Theorem (Niefield and Rosenthal)

The following are equivalent or a locale L:

(1) *L is totally spatial;*

(2) *$\mathcal{S}(L)^{op}$ is spatial.*

Let's say a frame L is **totally T_D-spatial** if every sublocale of L is T_D-spatial.

Theorem

The following are equivalent or a locale L:

(1) *L is totally T_D-spatial;*

(2) *$\mathcal{S}_D(L)^{op}$ is (T_D-)spatial.*

A few more equivalent conditions:

Theorem

The following are equivalent for a locale L.

(1) *L is totally T_D-spatial;*

(2) *$\mathcal{S}_D(L)^{op}$ is (T_D-)spatial;*

(3) *$\mathcal{S}_D(L)^{op}$ is (T_D-)spatial and Boolean (i.e. a complete and atomic Boolean algebra);*

(4) *All D-sublocales of L are T_D-spatial, that is, $\mathfrak{M} \circ \mathrm{pt}_D = 1_{\mathcal{S}_D(L)}$;*

(5) *$\mathcal{S}_D(L) = \mathcal{S}_b(L)$ and L is T_D-spatial;*

(6) *Every nonzero sublocale of L contains a covered prime in itself.*

Example

The Alexandroff topology on the natural numbers is totally T_D-spatial. Moreover, $\mathcal{S}_D(\Omega(\mathbb{N})) \subsetneq \mathcal{S}(\Omega(\mathbb{N}))$.

Functoriality

The assignment $L \mapsto \mathcal{S}_D(L)^{op}$ cannot be made functorial in Frm in such a way that there is a natural transformation $\mathfrak{c}\colon 1_{\mathsf{Frm}} \to \mathcal{S}_D(-)^{op}$. Therefore, we have to deal with *lifts* of individual frame homomorphisms, i.e. commutative squares in Frm of the form

$$\begin{array}{ccc}
\mathcal{S}_D(L)^{op} & \xrightarrow{\mathcal{S}_D(f)} & \mathcal{S}_D(M)^{op} \\
{\scriptstyle \mathfrak{c}_L}\big\uparrow & & \big\uparrow{\scriptstyle \mathfrak{c}_M} \\
L & \xrightarrow{\quad f \quad} & M
\end{array}$$

Natural question: is it a functor at least on Frm_D?

Proposition (Necessary condition)

Let $f : L \to M$ be a frame homomorphism. If f lifts then it is a D-homomorphism.

Proposition

Let L a frame and $f : L \twoheadrightarrow S$ a surjection onto a sublocale S. Then f lifts if and only if it is a D-homomorphism (i.e. iff S is a D-sublocale of L).

Example

For monomorphisms situation much worse: there is an open D-homomorphism between spatial locales which is a monomorphism and which does **not** lift.

T_D-points vs T_1-points

A couple of differences:

- Consider $\mathcal{S}_M(L)$ — the set of sublocales S such that $\max(S) \subseteq \max(L)$. It is **never** a subcolocale of $\mathcal{S}(L)$ if L contains a non-maximal covered prime.
- A T_1-locale (Rosický & Šmarda) is one in which primes are maximal. They are reflective in Loc.Locales in which all primes are covered (what should be called ***T_D*-locales**) do **not** form a reflective subcategory.

Relations between subcolocales of $\mathcal{S}(L)$

We have considered several subcolocales of the $\mathcal{S}(L)$:

- The Booleanization $\mathcal{S}_b(L)$;
- the spatialization $\mathrm{sp}[\mathcal{S}(L)]$;
- the coframe of D-sublocales $\mathcal{S}_D(L)$;

Recently, another subset has also enjoyed special attention:

- The frame of joins of closed sublocales $\mathcal{S}_c(L)$ (J. Picado, A. Pultr, A. Tozzi).

Relations between subcolocales of $\mathcal{S}(L)$

SUBSETS OF $\mathcal{S}(L)$	PROPERTY OF L
$\mathcal{S}(L) = \mathsf{sp}[\mathcal{S}(L)]$	Totally spatial
$\mathcal{S}_b(L) \subseteq \mathsf{sp}[\mathcal{S}(L)]$	Spatial
$\mathcal{S}_c(L) \subseteq \mathsf{sp}[\mathcal{S}(L)]$	Spatial
$\mathsf{sp}[\mathcal{S}_b(L)] = \mathcal{S}_b(L)$	T_D-spatial
$\mathcal{S}_b(L) = \mathsf{sp}[\mathcal{S}(L)]$	Strongly T_D-spatial
$\mathcal{S}_D(L) = \mathcal{S}(L)$	T_D-locale
$\mathsf{sp}[\mathcal{S}(L)] \subseteq \mathcal{S}_D(L)$	T_D-locale
$\mathsf{sp}[\mathcal{S}(L)] \subseteq \mathcal{S}_b(L)$	T_D-locale
$\mathsf{sp}[\mathcal{S}(L)] \subseteq \mathcal{S}_c(L)$	T_1-locale
$\mathcal{S}_D(L) \subseteq \mathsf{sp}[\mathcal{S}(L)]$	Totally spatial
$\mathcal{S}_D(L) = \mathsf{sp}[\mathcal{S}_D(L)]$	Totally T_D-spatial
$\mathcal{S}_b(L) = \mathcal{S}(L)$	Scattered
$\mathcal{S}_c(L) = \mathcal{S}(L)$	Scattered and fit
$\mathcal{S}_D(L) = \mathcal{S}_b(L)$	D-scattered
$\mathcal{S}_D(L) = \mathcal{S}_c(L)$	D-scattered and subfit

Thank you!