κ-hollow frames, κ-replete **W**-objects

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Frames and κ-frames

- The symbols λ and κ stand for infinite regular cardinal numbers, with the understanding that $\omega < \lambda \leq \kappa$.
- A *κ-set* is a set C such that $|C| < \kappa$, and a *κ-subset* is a subset $C \subseteq A$ which is a κ-set; we write $C \subseteq_\kappa A$. A *κ-join* in a lattice is the join of a κ-set.
- A *κ-frame* is a bounded distributive lattice L which is closed under finite meets and κ-joins, in which finite meets distribute over κ-joins: $a_0 \wedge \bigvee A = \bigvee (A \wedge a_0)$, $A \subseteq_\kappa L$.
- A κ-frame homomorphism is a mapping between κ-frames which preserves κ-joins (including the empty join $\bot$) and finite meets (including the empty meet $\top$).
- When no cardinal bound is mentioned, none applies.
- A frame is $\ldots$.

Complete regularity

- In a κ-frame L, we say that *a is rather below b*, and write $a \prec b$, if there exists some $c \in L$ such that $a \wedge c = \bot$ and $b \vee c = \top$. We say that *a is completely below b*, and write $a \prec\!\!\prec b$, if there exists a family $\{c_q : q \in \mathbb{Q}\} \subseteq L$ such that $a \leq c_p \prec c_q \leq b$ for all $p < q$ in $\mathbb{Q}$.
- A κ-frame L is said to be *completely regular* if each element is a κ-join of elements completely below it. *We assume all frames and all κ frames are completely regular, and all spaces are Tychonoff.*
- We denote the category of κ-frames with their homomorphisms by $\boldsymbol{\kappa}$**Frm**, and the category of frames with their homomorphisms by **Frm**.

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- Every frame is a κ-frame for sufficiently large κ, but....

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- Proposition

  κFrm is coreflective in **Frm**, i.e., every frame L has a largest sub-κ-frame, designated L_κ, and every frame homomorphism $L \to M$ drops to a unique κ-frame homomorphism $L_\kappa \to M_\kappa$.

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  - The elements of L_κ are called *κ-cozero elements*, or simply *κ-cozeros*. L_κ itself is called the *κ-cozero part of* L.

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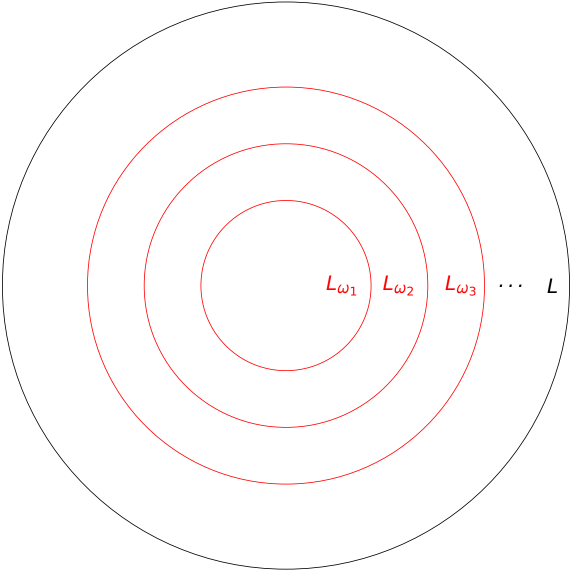
  - The elements of L_κ are called *κ-cozero elements*, or simply *κ-cozeros*. L_κ itself is called the *κ-cozero part of L*.
  - It is characteristic of completely regular frames that the cozero elements join generate the frame.

The ascending tower of κ-cozero parts of L



The ascending tower of κ-cozero parts of L

$$
\begin{array}{c}
L_{\omega_1} \\
\downarrow \\
L_{\omega_2} \\
\downarrow \\
L_{\omega_3} \\
\vdots \\
\downarrow \\
L
\end{array}
$$

The free frame over a κ-frame

There is a free frame over any κ-frame.

Proposition (Madden 1991)

For any κ-frame L, the set $\mathcal{I}L$ of κ-ideals of L is a frame, and the map $L \to \mathcal{I}L = (a \mapsto \downarrow a \downarrow)$ is a κ-frame isomorphism $L \to (\mathcal{I}L_\kappa)_\kappa$ which functions as the free frame over L, in the sense that any κ-frame homomorphism from L into a frame M factors through the map $L \to \mathcal{I}L$.

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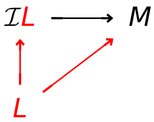

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An element a in a frame L is called *κ-Lindelöf*, or simply a *κ-element*, if whenever $\bigvee A \geq a$ for some $A \subseteq L$ there exists $A_0 \subseteq_\kappa A$ such that $\bigvee A_0 \geq a$. The frame L itself is called κ-Lindelöf if $\top$ is a κ-element.

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Theorem (Madden 1991)

The following are equivalent for a frame L.

1. The set E of κ-elements of L forms a join-generating sub-κ-frame of L.
2. E is closed under finite meets and join generates M.
3. L is κ-Lindelöf.
4. L is κ-free, i.e., isomorphic to $\mathcal{I}E$.

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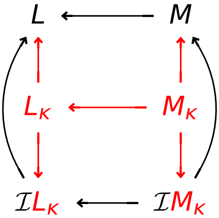
The κ-Lindelöf coreflection in **Frm**

Proposition

The full subcategory $\boldsymbol{\kappa}$**LFrm** of κ-Lindelöf frames is coreflective in **Frm**. A coreflector for the frame L is the map $\mathcal{I}L_\kappa \to L = (I \mapsto \bigvee I)$.



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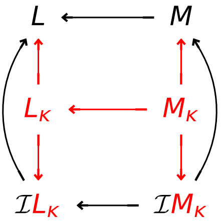

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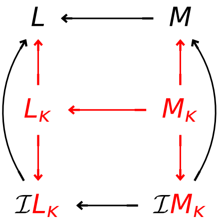

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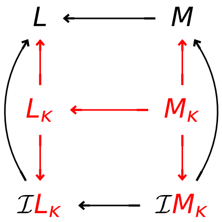

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Dense **Frm** and ***κ*****Frm** morphisms

A (**Frm**- or ***κ*****Frm**-)morphism $m \colon L \to M$ is said to be *dense* if $m(a) = \bot$ implies $a = \bot$.

Proposition

A morphism in **Frm** or ***κ*****Frm** is monic iff it is dense.

Theorem (Isbell for **Frm**, Madden for ***κ*****Frm**)

Every object L of either **Frm** or ***κ*****Frm** has a smallest dense quotient corresponding to the *codensity congruence* Δ:

$$(a, b) \in \Delta \iff \forall c\, (c \wedge a = \bot \iff c \wedge b = \bot).$$

We refer to L/Δ as *the skeleton of* L, and we denote it and its quotient map by

$$s \colon L \to \mathcal{S}L.$$

To reiterate, s is a dense surjection which factors through every dense surjection out of L.

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Note that a one-one morphism is dense, but not conversely.

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Essential completions in **Frm** and **κFrm**

In algebra, an extension $A \subseteq B$ is *essential* if any nontrivial collapse of B results in a nontrivial collapse of A.

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The skeleton map $s\colon L \to \mathcal{S}L$ is the essential completion in either **Frm** or **κFrm**. Neither category has injectives.

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In the category **nFrm** of naked frames, monos are one-one, and the essential completion of a frame L is its embedding into the skeleton of its congruence lattice. This category also lacks injectives.

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In the category ℓ**Grp** of ℓ-groups, every object G has a unique maximal extension which is essential in the category of distributive lattices.

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A corollary of the density theorems

Corollary

A λ-Lindelöf frame L has a smallest dense κ-Lindelöf quotient, designated $t_\kappa \colon L \to \mathcal{M}_\kappa L$.

Corollary

An ω_1-Lindelöf frame L has a descending cardinally indexed tower of minimal κ-Lindelöf quotients, beginning with $\mathcal{M}_{\omega_1} L$ and ending with $\mathcal{S}L$.

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Proof

Since **κLFrm** is coreflective, it is closed under intersection, so the intersection of the dense κ-Lindelöf quotients of L exists. It is dense by Isbel's density theorem. □

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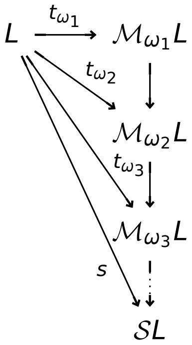

Skeletal **κFrm**-morphisms

A **κFrm**-morphism $m\colon L \to M$ is called *skeletal* if it takes dense elements of L to dense elements of M.

Lemma

A **κFrm**-morphism $m\colon L \to M$ is skeletal iff it drops to the skeletons.

$$\begin{array}{ccc} L & \xrightarrow{m} & M \\ \downarrow{\scriptstyle s_L} & & \downarrow{\scriptstyle s_M} \\ \mathcal{S}L & \longrightarrow & \mathcal{S}M \end{array}$$

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About κ-frame skeletons

Proposition

The following are equivalent for a κ-frame L.

- The codensity congruence Δ is the identity on L, i.e., for all $a, b \in L$, $a^\perp = b^\perp$ implies $a = b$, where $a^\perp \equiv \{ c : a \wedge c = \perp \}$.
- L has no proper dense quotients.
- Every mono out of L is one-one.
- The skeleton map $s : L \to \mathcal{S}L$ is one-one, hence a κ-frame isomorphism.
- L is isomorphic to an order dense sub-κ-frame of a complete boolean algebra.
- L has no proper dense elements.
- L is *separative,* i.e., for all $a \nleq b$ there exists c such that $\perp < c \leq a$ and $c \wedge b = \perp$.
- Every κ-frame homomorphism out of L is skeletal.

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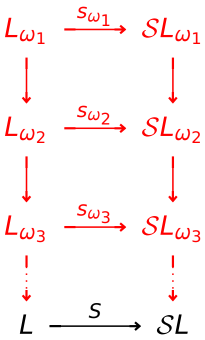
- L is isomorphic to an order dense sub-κ-frame of a complete boolean algebra.
- L has no proper dense elements.
- L is *separative,* i.e., for all $a \not\leq b$ there exists c such that $\perp < c \leq a$ and $c \wedge b = \perp$.
- Every κ-frame homomorphism out of L is skeletal.

We call a κ-frame with these attributes *hollow*.

A second look at the tower of κ-cozero parts of an ω_1-Lindelöf frame L

- Start with the tower of κ-cozero parts of an ω_1-Lindelöf frame in the left column. Then add in their skeletons in the right column.
- The vertical arrows are all injections, the horizontal arrows are all surjections.
- The members of L_κ are the κ-joins of members of L_{ω_1}, and similarly for the members of $\mathcal{S}L_\kappa$.
- Therefore the whole diagram is generated by L_{ω_1}.



A second look at the tower of κ-cozero parts of an ω_1-Lindelöf frame L

- Start with the tower of κ-cozero parts of an ω_1-Lindelöf frame in the left column. Then add in their skeletons in the right column.
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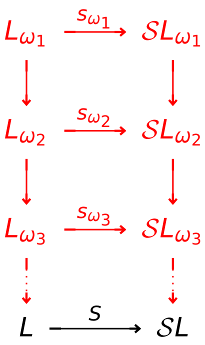

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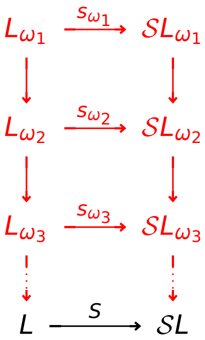

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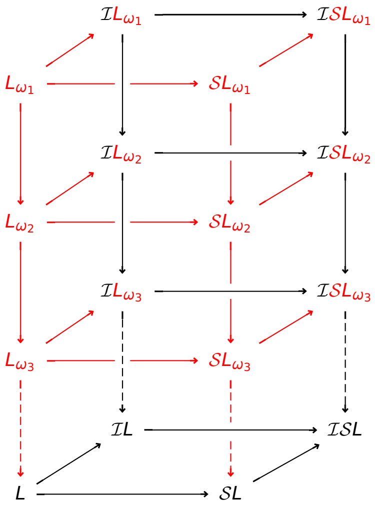
$$
\begin{array}{ccc}
L_{\omega_1} & \xrightarrow{s_{\omega_1}} & \mathcal{S}L_{\omega_1} \\
\downarrow & & \downarrow \\
L_{\omega_2} & \xrightarrow{s_{\omega_2}} & \mathcal{S}L_{\omega_2} \\
\downarrow & & \downarrow \\
L_{\omega_3} & \xrightarrow{s_{\omega_3}} & \mathcal{S}L_{\omega_3} \\
\vdots & & \vdots \\
L & \xrightarrow{s} & \mathcal{S}L
\end{array}
$$

Now apply the free frame functor to the diagram



What is the top layer?

Theorem

In a λ-Lindelöf frame L, the minimal dense κ-Lindelöf quotient is (isomorphic to) the free frame over the skeleton of the κ-cozero part of L. In symbols, $\mathcal{I}\mathcal{S}L_\kappa \approx \mathcal{M}_\kappa L$.

$$
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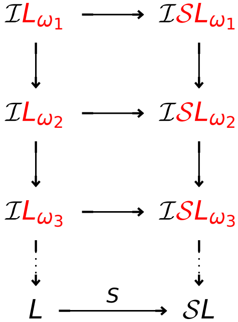

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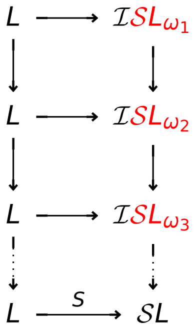

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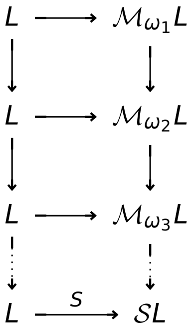

Hollowing out a frame

The skeleton $\mathcal{S}L_{\omega_1}$ of L_{ω_1} has no dense elements, so the same is true of the free frame $\mathcal{I}L_{\omega_1}$ over it. By the theorem, then, $\mathcal{M}_{\omega_1}L$ has no dense ω_1-cozero elements.

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L & \longrightarrow & \mathcal{M}_{\omega_1}L \\
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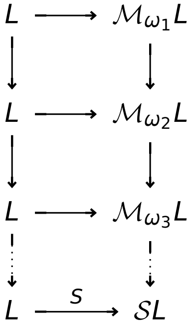

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Eventually, the skeleton $\mathcal{S}L$ has no dense elements at all, i.e., it is a complete boolean algebra.

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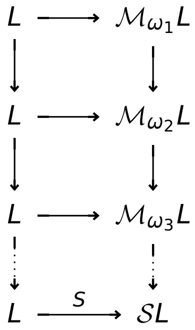
We view this as a process of hollowing out the frame by successively removing dense κ-cozeros in increasing order of their complexity.



κ-hollow frames

Theorem

The following are equivalent for a λ-Lindelöf frame L.

- L_κ is hollow.
- L has no dense κ-cozero elements.
- $a = a^{**}$ for every κ-cozero element $a \in L$.
- For all $a, b \in L$, if $a \leq b$ and a is a κ-cozero then $a^{**} \leq b$.
- For all $a, b \in L$, if $a \wedge d = b \wedge d$ for every dense κ-cozero element d then $a = b$.
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Dube has provided a beautiful characterization of ω_1-hollow

κ-skeletal frame homomorphisms

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The following are equivalent for a frame homomorphism $m\colon L \to M$.

- m takes dense κ-cozero elements of L to dense κ-cozero elements of M.
- m drops to a skeletal κ-frame morphism $L_\kappa \to M_\kappa$.
- m drops to a κ-frame morphism $\mathcal{S}L_\kappa \to \mathcal{S}M_\kappa$.
- $m(a^{**}) \leq m(a)^{**}$ for all $a \in L_\kappa$.
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Theorem

κhFrm is epireflective in **Frmκsk**, and a reflector for L is the map which takes L to the intersection of its dense κ-cozero quotients. This coincides with $\mathcal{M}_\kappa L$ if L is κ-Lindelöf.

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How do κ-hollow frames interact with **W**?

Theorem (Madden and Vermeer)

$\mathcal{R}\colon \mathbf{Frm} \to \mathbf{W}$ is adjoint. That is, for any **W**-object G there is a frame L and a **W**-injection $\mu_G\colon G \to \mathcal{R}L$ which is universal among such arrows.

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For a given **W**-object G, the family

$$\mathcal{K}G \equiv \{ K : K \text{ is a } \mathbf{W}\text{-kernel of } G \}$$

forms an ω_1-Lindelöf frame under inclusion. In fact, $\mathcal{K}\colon \mathbf{W} \to \mathbf{Frm}$ is a functor.

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More about **W**

We denote by $[K]$ the **W**-kernel generated by a subset $K \subseteq G$.

Lemma

Let $K \in \mathcal{K}G$ be a **W**-kernel of a **W**-object G.

1. K is a κ-cozero element of $\mathcal{K}G$ iff K is κ-generated.
2. The **W**-quotient map $G \to G/K$ is κ-complete, i.e., preserves κ-joins, iff K is κ-order closed, i.e., closed under κ-joins.
3. K is order closed, i.e., closed under all joins, iff K is a polar.
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κ-complete **W**-homomorphisms

Proposition

The following are equivalent for a **W**-homorphism $\theta\colon G \to H$.

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5. $\mathcal{K}\theta$ is κ-skeletal, where $\mathcal{K}\theta\colon \mathcal{K}G \to \mathcal{K}H$ realizes θ.

Theorem

$\mathcal{R}\colon \boldsymbol{\kappa}\mathbf{hFrm} \to \mathbf{W}\boldsymbol{\kappa}$ is adjoint. That is, for any $\mathbf{W}\boldsymbol{\kappa}$-object G there is a κ-hollow frame M and a κ-complete injection $\gamma_G\colon G \to \mathcal{R}M$ which is universal among all such arrows.

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Replete **W**-objects

Proposition

The following are equivalent for a **W**-object G.

1. Every κ-generated **W**-kernel is a polar.
2. Every proper κ-generated **W**-kernel is contained in a proper polar.
3. $K_1 \cap K_2^{\perp} \neq 0$ for κ-generated **W**-kernels K_i such that $K_1 \not\subseteq K_2$.
4. For **W**-kernels $K_i \subseteq G$, if $K_1 \subseteq K_2$ and K_1 is κ-generated then $K_1^{\perp\perp} \subseteq K_2$.
5. $\mathcal{K}G$ is κ-hollow.
6. Every **W**-homomorphism out of G is κ-complete.
7. Every **W**-kernel of G is κ-closed.

Replete **W**-objects

Proposition

The following are equivalent for a **W**-object G.

1. Every κ-generated **W**-kernel is a polar.
2. Every proper κ-generated **W**-kernel is contained in a proper polar.
3. $K_1 \cap K_2^{\perp} \neq 0$ for κ-generated **W**-kernels K_i such that $K_1 \nsubseteq K_2$.
4. For **W**-kernels $K_i \subseteq G$, if $K_1 \subseteq K_2$ and K_1 is κ-generated then $K_1^{\perp\perp} \subseteq K_2$.
5. $\mathcal{K}G$ is κ-hollow.
6. Every **W**-homomorphism out of G is κ-complete.
7. Every **W**-kernel of G is κ-closed.

A **W**-object G which has the properties listed above is called *κ-replete*. A *κ-repletion of G* is a **W**-essential extension which is replete, and a *minimal κ-repletion* is a κ-repletion which is an initial factor of every κ-repletion of G.

κ-projectable **W**-objects

A **W**-object G is said to be *κ-projectable* if each κ-generated polar is a cardinal summand.

Proposition

Every **W**-object G has a unique *κ-projectable hull*, i.e., an essential extension $G \to H$ such that H is κ-projectable, but no proper initial factor of the extension is κ-projectable.

Theorem

The following are equivalent for $G = \mathcal{R}L$.

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2. G is κ-projectable.
3. L is κ-hollow.

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Thank you.

Thank you very much.