Localic maps and z-embedded sublocales

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Part 1

Background and Notation

Frm and Loc

- The category of frames **Frm**:
  - A frame L is a complete lattice in which

    $$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

    for any $a \in L$ and $S \subseteq L$. (L is also a complete Heyting algebra. It has a Heyting operator denoted by $\to$.)
  - A map $L \to M$ is a frame homomorphism if it preserves arbitrary joins and finite meets.
- **Loc=Frm**^{op}
- Localic maps can be seen as the right adjoints of frame homomorphisms. In fact,
- $f : L \to M$ is a localic map if:
  (1) it preserves arbitrary meets,
  (2) $f(a) = 1 \Rightarrow a = 1$, and
  (3) $f(f^*(a) \to b) = a \to f(b)$ where $L \underset{f}{\overset{f^*}{\rightleftarrows}} M$ ($f^* \dashv f$).

$\mathcal{S}(L)^{op}$

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- $S \subseteq L$ is a sublocale of $L \Leftrightarrow S$ is a locale with the induced order of L and the embedding $j\colon S \hookrightarrow L$ is a localic map.
- The system $\mathcal{S}(L)$ of all sublocales of a locale L, partially ordered by inclusion, is a coframe (that is, $\mathcal{S}(L)^{op}$ is a frame).
  - $\bigwedge_{i \in J} S_i = \bigcap_{i \in J} S_i$
  - $\bigvee_{i \in J} S_i = \{\bigwedge M \mid M \subseteq \bigcup_{i \in J} S_i\}$
  - Least element: $0 = \{1\}$. Greatest element: $1 = L$.

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Being a co-Heyting algebra, $\mathcal{S}(L)$ has co-pseudocomplements that we denote by $S^{\#}$.

open and closed sublocales

For any $a \in L$, the sublocales

$$\mathfrak{c}_L(a) = \uparrow a = \{x \in L \mid x \geq a\} \text{ and } \mathfrak{o}_L(a) = \{a \to b \mid b \in L\}$$

are the **closed** and **open** sublocales of L, respectively.

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- $\mathfrak{c}(L)$ is a subcoframe of $\mathcal{S}(L)$ (that is, $\mathfrak{c}(L)^{op}$ is a subframe of $\mathcal{S}(L)^{op}$).
- $$L \xrightarrow{\cong} \mathfrak{c}(L)^{op} \hookrightarrow \mathcal{S}(L)^{op}$$

The frame of Reals

Recall the frame of reals $\mathcal{L}(\mathbb{R})$. We define it as the frame presented by:

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  (r3) $(p, -) = \bigvee_{r>p}(r, -)$,
  (r4) $(-, q) = \bigvee_{s<q}(-, s)$,
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For each $p \leq q$ in $\mathbb{Q}$, the element $(p, -) \wedge (-, q)$ in $\mathcal{L}(\mathbb{R})$ is denoted by (p, q).

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$\Sigma\mathcal{L}(\mathbb{R})$ and $\mathbb{R}$ are homeomorphic, and $\mathcal{L}(\mathbb{R})$ and $\mathcal{O}\mathbb{R}$ are isomorphic frames.

Real-valued functions

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- A **general real-valued function** on L is a frame homomorphism $\mathcal{L}(\mathbb{R}) \to \mathcal{S}(L)^{op}$. We denote by $\mathrm{F}(L)$ the ring of all real-valued functions on L.
- A real function $f\colon \mathcal{L}(\mathbb{R}) \to \mathcal{S}(L)^{op}$ is **continuous** if $f(p,-)$ and $f(-,q)$ are closed sublocales for every $p,q \in \mathbb{Q}$. We denote by $\mathrm{C}(L)$ the subring of $F(L)$ given by all the continous functions of L.

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- A real function $f\colon \mathcal{L}(\mathbb{R}) \to \mathcal{S}(L)^{op}$ is **continuous** if $f(p,-)$ and $f(-,q)$ are closed sublocales for every $p,q \in \mathbb{Q}$. We denote by $\mathsf{C}(L)$ the subring of $F(L)$ given by all the continous functions of L.

- Note that $\mathsf{C}(L)$ is isomorphic to $\mathcal{R}(L)$ (since

$$L \xrightarrow{\cong} \mathfrak{c}(L)^{op} \hookrightarrow \mathcal{S}(L)^{op}\ ).$$

Image and Preimage

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- $f[S]$ is a sublocale of M for any sublocale $S \subseteq L$.

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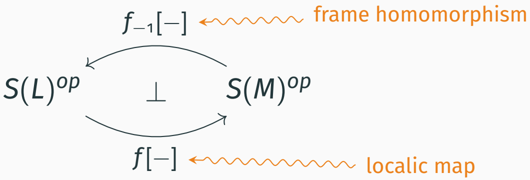
- There is an adjunction: $f[S] \subseteq T$ iff $S \subseteq f_{-1}[T]$.



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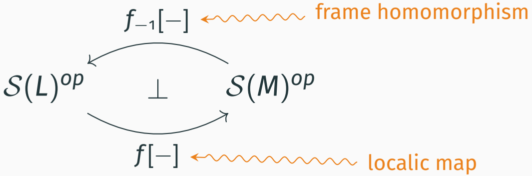

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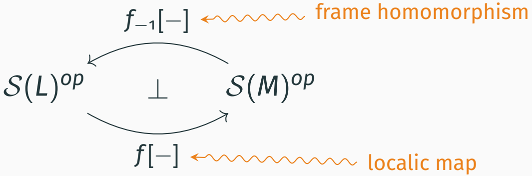
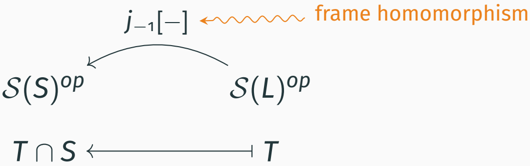


In particular when S is a sublocale of L:

- The embedding $j\colon S \hookrightarrow L$ is a localic map, and



Part 2

Zero Sublocales

Zero sublocales

- A zero sublocale of L is a sublocale of the form $f(0,-) \cap f(-,0)$ for some continuous $f\colon \mathcal{L}(\mathbb{R}) \to \mathcal{S}(L)^{op}$. We denote by $\mathsf{ZS}(L)$ the σ-coframe of all zero sublocales of L.

Zero sublocales

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- A cozero sublocale of L is a sublocale of the form $(f(0,-) \vee f(-,0))^{\#}$ for some continuous $f\colon \mathcal{L}(\mathbb{R}) \to \mathcal{S}(L)^{op}$. We denote by CoZS($L$) the σ-frame of all cozero sublocales of L.

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- A cozero sublocale of L is a sublocale of the form $(f(0,-) \vee f(-,0))^{\#}$ for some continuous $f\colon \mathcal{L}(\mathbb{R}) \to \mathcal{S}(L)^{op}$. We denote by $\mathsf{CoZS}(L)$ the σ-frame of all cozero sublocales of L.

- $Coz(L) \cong \mathsf{ZS}(L)^{op} \cong \mathsf{CoZS}(L)$ as σ-frames.

Zero sublocales and classical motivation

cozero element:	$f(0,-) \vee f(-,0)$	$f\colon \mathcal{L}(\mathbb{R}) \to L$
zero sublocale:	$\mathfrak{c}(a) \quad a \in Coz(L)$	
	$f(0,-) \cap f(-,0)$	$f\colon \mathcal{L}(\mathbb{R}) \to \mathcal{S}(L)^{op}$
cozero sublocale:	$\mathfrak{o}(a) \quad a \in Coz(L)$	
	$(f(0,-) \cap f(-,0))^{\#}$	$f\colon \mathcal{L}(\mathbb{R}) \to \mathcal{S}(L)^{op}$

Zero sublocales and classical motivation

cozero element: $f(0,-) \vee f(-,0) \qquad f\colon \mathcal{L}(\mathbb{R}) \to L$

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zero sublocale: $\mathfrak{c}(a) \quad a \in Coz(L)$

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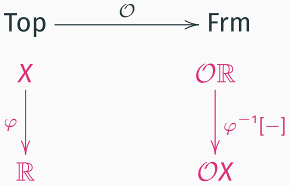

- $f\colon \mathcal{L}(\mathbb{R}) \to L$

  $f(p,-) \sim \varphi^{-1}(p,\infty)$

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zero sublocale:	$\mathfrak{c}(a) \quad a \in Coz(L)$	
	$f(0,-) \cap f(-,0)$	$f\colon \mathcal{L}(\mathbb{R}) \to \mathcal{S}(L)^{op}$
	$\varphi^{-1}(-\infty,0] \cap \varphi^{-1}[0,\infty) = \varphi^{-1}(\{0\})$	
cozero sublocale:	$\mathfrak{o}(a) \quad a \in Coz(L)$	
	$(f(0,-) \cap f(-,0))^{\#}$	$f\colon \mathcal{L}(\mathbb{R}) \to \mathcal{S}(L)^{op}$
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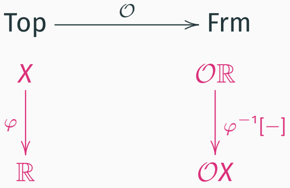

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  $f(p,-) \sim \varphi^{-1}(-\infty,p]$

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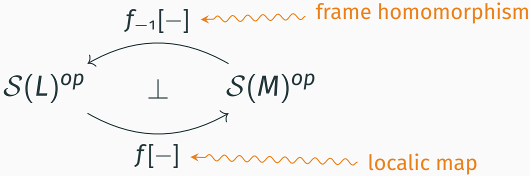
Part 3

z-maps

z-maps

Let $f\colon L \to M$ be a localic map



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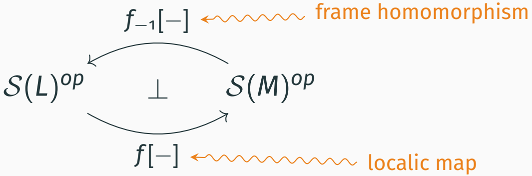

We can restrict the preimage frame homomorphism to the zero sublocales. Then

$$f^{z}_{-1}[-]\colon ZS(M)^{op} \to ZS(L)^{op}$$

is a well-defined σ-frame homomorphism.

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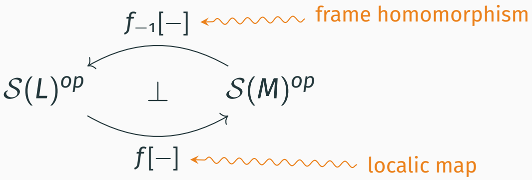

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$$f^{z}_{-1}[-]\colon \mathsf{ZS}(M)^{op} \to \mathsf{ZS}(L)^{op}$$

is a well-defined σ-frame homomorphism. We say f is a ***z*-map** if $f^{z}_{-1}[-]$ is surjective.

z-embedded sublocales

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z-embedded sublocales

- When the embedding of a sublocale $j\colon S \hookrightarrow L$ is a z-map, we say S is ***z*-embedded** in L.

- S is z-embedded in L if and only if

$$j_{-1}[-]\colon \mathsf{ZS}(L)^{op} \to \mathsf{ZS}(S)^{op}$$
$$j_{-1}[T] = T \cap S$$

  is surjective if and only if for every zero sublocale Z of S there is a zero sublocale W of L such that $Z = W \cap S$.

Examples of z-embedded sublocales

- C-embedded $\Rightarrow$ C^*-embedded $\Rightarrow$ z-embedded.

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- C-embedded $\Rightarrow$ C^*-embedded $\Rightarrow$ z-embedded.
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- If L is normal every closed sublocale is z-embedded.

- If L is normal every F_σ-sublocale is z-embedded. S is an F_σ-sublocale if $S = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(a_n)$.

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- If L is normal every F_σ-generalized sublocale is z-embedded. S is F_σ-generalized if whenever $S \subseteq \mathfrak{o}(a)$ there is an F_σ-sublocale F such that $S \subseteq F \subseteq \mathfrak{o}(a)$.

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- If L is mildly normal every closed regular sublocale is z-embedded. S is closed regular if $S = \mathfrak{c}(a^*)$ for some $a \in L$.

Characterization of normality

Theorem

The following statements about a locale L are equivalent.

(1) L is normal.
(2) Every closed sublocale of L is C-embedded in L.
(3) Every closed sublocale of L is C^*-embedded in L.
(4) Every closed sublocale of L is z-embedded in L.
(5) Every F_σ-sublocale of L is z-embedded in L.
(6) Every generalized F_σ-sublocale of L is z-embedded in L.

Characterization of mild normality

Theorem

The following statements about a locale L are equivalent.

(1) L is mildly normal.

(2) Every regular closed sublocale of L is C^*-embedded in L.

(3) Every regular closed sublocale of L is z-embedded in L.

C-, C^*- and z-maps

Theorem

Let L be a locale. The following assertions are equivalent:

(1) Every closed localic map with codomain L is a C-map.
(2) Every closed localic map with codomain L is a C^*-map.
(3) Every closed localic map with codomain L is a z-map.

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