Maximal $d$-Elements of $M$-frames

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BLAST 2021

June 10, 2021
Preliminaries
Primes and Minimal Primes
Max($dL$)
The space $Max(dL)$
Ultrafilters of $R^L$
Definition

1. A frame $L$ is a complete lattice that satisfies a strong distributive law where finite infimum distributes over arbitrary supremum.

2. An element $c \in L$ is compact if $c \leq \bigvee a_\alpha$ implies that $c \leq a_{\alpha_1} \lor \cdots \lor a_{\alpha_n}$. The collection of all compact element of $L$ is denoted by $\mathcal{R}(L)$.

3. A frame is algebraic if every element in the frame is the supremum of compact elements.

4. A frame is said to satisfy the finite intersection property (FIP) if $c, d \in \mathcal{R}(L)$ implies that $c \land d \in \mathcal{R}(L)$.

5. For each $x \in L$, $x^\perp = \bigvee \{y \in L : y \land x = 0\}$.

6. $x \in L$ is dense if $x^\perp = 0$. A compact, dense element of $L$ is called a unit.

All the frames are $M$-frames (algebraic and satisfies the FIP), and possesses a unit.
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All the frames are \(M\)-frames (algebraic and satisfies the FIP), and possesses a unit.
- An element \( p \in L \) is *prime* if \( p < 1 \) and for all \( a, b \in L \), \( a \wedge b \leq p \) implies that \( a \leq p \) or \( b \leq p \).

- A prime element \( p \) is *minimal* if there are no other prime elements \( q < p \). We denote the collection of minimal prime elements of \( L \) by \( \text{Min}(L) \). Using Zorn’s lemma we can show that primes and minimal primes exist in algebraic frames.

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\text{Spec}(L) = \{\text{all primes of } L\}
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- An element $p \in L$ is **prime** if $p < 1$ and for all $a, b \in L$, $a \land b \leq p$ implies that $a \leq p$ or $b \leq p$.

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$Min(L) = \{\text{all minimal primes of } L\}$
- **Zariski Topology on** \( \text{Min}(L) \):  
Let \( c \in L \).  
Define \( U(c) = \{ p \in \text{Min}(L) : c \not\preceq p \} \)

**Lemma**

Let \( x, y \in L \). The following holds:

1. \( U(x) \cup U(y) = U(x \lor y) \) and \( U(x) \cap U(y) = U(x \land y) \).
2. \( \bigcup U(x_\alpha) = U \left( \bigvee x_\alpha \right) \).
3. \( U(x) = \emptyset \iff x = 0 \).

The **Zariski topology** on \( \text{Min}(L) \) is the topology generated by the collection \( B = \{ U(c) \mid c \in \mathcal{R}(L) \} \).

**Lemma**

\( \text{Min}(L) \) is a Hausdorff, zero-dimensional (base of clopen sets) space.
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$\text{Min}(L)$ is a Hausdorff, zero-dimensional (base of clopen sets) space.
Question: When is $Min(L)$ a compact space?


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- **Inverse Topology on** \( \text{Min}(L) \), **write as** \( \text{Min}(L)^{-1} \):

Define the set-theoretic complement of \( U(x) \), called \( V(x) = \{ p \in \text{Min}(L) : x \leq p \} \). The following holds for the operator \( V(\cdot) \):

- \( V(x) \cup V(y) = V(x \land y) \) and \( V(x) \cap V(y) = V(x \lor y) \).
- \( V(x) = \text{Min}(L) \iff x = 0. \)

The collection \( \{ V(k) : k \in \mathcal{R}(L) \} \) forms a basis for a topology on \( \text{Min}(L) \), called the **Inverse topology**, and denoted by \( \text{Min}(L)^{-1} \).

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**Lemma**

\( \text{Min}(L)^{-1} \) is a compact, \( T_1 \) space.
- **Inverse Topology on** $\text{Min}(L)$, **write as** $\text{Min}(L)^{-1}$:

Define the set-theoretic complement of $U(x)$, called $V(x) = \{p \in \text{Min}(L) : x \leq p\}$. The following holds for the operator $V(\cdot)$.

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- \( V(x) \cup V(y) = V(x \wedge y) \) and \( V(x) \cap V(y) = V(x \vee y) \).
- \( V(x) = \text{Min}(L) \Leftrightarrow x = 0 \).

The collection \( \{ V(k) : k \in \mathcal{K}(L) \} \) forms a basis for a topology on \( \text{Min}(L) \), called the **Inverse topology**, and denoted by \( \text{Min}(L)^{-1} \).

**Lemma**

\( \text{Min}(L)^{-1} \) is a compact, \( T_1 \) space.
- **Inverse Topology on** $\text{Min}(L)$, **write as** $\text{Min}(L)^{-1}$:

Define the set-theoretic complement of $U(x)$, called $V(x) = \{ p \in \text{Min}(L) : x \leq p \}$. The following holds for the operator $V(\cdot)$.

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The collection $\{ V(k) : k \in \mathcal{R}(L) \}$ forms a basis for a topology on $\text{Min}(L)$, called the **Inverse topology**, and denoted by $\text{Min}(L)^{-1}$.

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Ultrafilters of \( \mathcal R L^\perp \)

\textbf{Theorem}

The following are equivalent for an \( M \)-frame \( L \).

- The Zariski topology on \( \text{Min}(L) \) is compact.
- \( \text{Min}(L) = \text{Min}(L)^{-1} \).
- For each \( x \in \mathcal R(L) \) there exists \( y \in \mathcal R(L) \) such that \( x \land y = 0 \) and \( x \lor y \) is a unit.
- \( L \) is a complemented frame.
Lemma on Ultrafilters

For an $M$-frame $L$, $\text{Min}(L)$ is in bijective correspondence with $\text{Ult}(\mathcal{K}(L))$. In particular,

$$p \in \text{Min}(L) \text{ implies } F_p = \{ c \in \mathcal{K}(L) : c \not\preceq p \} \in \text{Ult}(\mathcal{K}(L)),$$

and

$$U \in \text{Ult}(\mathcal{K}(L)) \text{ implies } p(U) = \bigvee \{ c^\perp : c \in U \} \in \text{Min}(L).$$

Additionally, $p(F_p) = p$ and $F_{p(U)} = U$.

Hence, the map $\Phi : \text{Ult}(\mathcal{K}(L)) \to \text{Min}(L)$ defined by $\Phi(U) = p(U)$ is a well-defined bijection with $\Phi^{-1} = F_p$. 
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Topologies on $\text{Ult}(\mathcal{K}(L))$:

Given $c \in \mathcal{K}L$, we define $\Upsilon(c) = \{U \in \text{Ult}(\mathcal{K}L) : c \notin U\}$ and $\Upsilon^c(c) = \{U \in \text{Ult}(\mathcal{K}L) : c \in U\}$.

- The **Wallman topology** on $\text{Ult}(\mathcal{K}(L))$ is generated by the collection $B_\Upsilon = \{\Upsilon(c) \mid c \in \mathcal{K}(L)\}$.
- The **inverse topology** on $\text{Ult}(\mathcal{K}(L))$, denoted $\text{Ult}(\mathcal{K}(L))^{-1}$, is generated by the collection $B_{\Upsilon^c} = \{\Upsilon^c(c) \mid c \in \mathcal{K}(L)\}$.

**Theorem**

Let $L$ be an $M$-frame with a unit. The following holds:

1. The topological space $\text{Ult}(\mathcal{K}L)$ is homeomorphic to $\text{Min}(L)^{-1}$.
2. The topological space $\text{Ult}(\mathcal{K}L)^{-1}$ is homeomorphic to $\text{Min}(L)$. 
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Topologies on $\text{Ult}(\mathcal{L})$:

Given $c \in \mathcal{L}$, we define

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- The **Wallman topology** on $\text{Ult}(\mathcal{L})$ is generated by the collection $B_{\Upsilon} = \{\Upsilon(c) \mid c \in \mathcal{L}\}$.

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Given $c \in \mathcal{K}L$, we define 

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Primes and Minimal Primes

Definition

1. An element \( d \in L \) is a \textit{d-element} if \( d = \bigvee \{ c \perp \perp : c \in \mathcal{K}(L), c \leq d \} \). The collection of all \textit{d-elements} of \( L \) is denoted by \( dL \).

2. \( x \in dL \) if and only if given any \( c \in \mathcal{K}(L), c \leq x \iff c \perp \perp \leq x \).

3. Given an \( x \in L \), \( x_d = \bigvee \{ c \perp \perp : c \leq x, c \in \mathcal{K}L \} \in dL \). This gives a nucleus \( d : L \to dL \) such that \( d(x) = x_d \), and \( \text{fix}(d) = dL \). If \( c \in \mathcal{K}L \), then \( c_d = c \perp \perp \), and \( x \leq x_d \) for all \( x \in L \).

- \( dL \) is a frame, where meet is same as in \( L \) and join is given by \( \bigvee^d S = d(\bigvee S) = (\bigvee S)_d \).

- If \( L \) is algebraic, then \( \text{fix}(d) = dL \) is algebraic. In this case, \( \mathcal{K}(dL) = d(\mathcal{K}L) = \{ c \perp \perp : c \in \mathcal{K}L \} \).

- When \( L \) satisfies the FIP, \( dL \) satisfies the FIP.
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Definition

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An element \( d \in L \) is a \( d \)-element if \( d = \bigvee \{ c^\perp : c \in \text{K}(L), c \leq d \} \). The collection of all \( d \)-elements of \( L \) is denoted by \( dL \).

\( x \in dL \) if and only if given any \( c \in \text{K}(L), c \leq x \iff c^\perp \leq x \).

Given an \( x \in L \), \( x_d = \bigvee \{ c^\perp : c \leq x, c \in \text{K}L \} \in dL \). This gives a nucleus \( d : L \rightarrow dL \) such that \( d(x) = x_d \), and \( \text{fix}(d) = dL \). If \( c \in \text{K}L \), then \( c_d = c^\perp \), and \( x \leq x_d \) for all \( x \in L \).

\(- dL \) is a frame, where meet is same as in \( L \) and join is given by \( \bigvee^d S = d(\bigvee S) = (\bigvee S)_d \).

\(- If \( L \) is algebraic, then \( \text{fix}(d) = dL \) is algebraic. In this case, \( \text{K}(dL) = d(\text{K}L) = \{ c^\perp : c \in \text{K}L \} \).

\(- When \( L \) satisfies the FIP, \( dL \) satisfies the FIP.
Existence of $\text{Max}(dL)$:
- Suppose $L$ is an $M$-frame that contains a unit $u$.
- Let $A = \{ x \in dL : x < 1 \}$. Obviously $0 \in A$, which means that $A \neq \emptyset$.
- Take a chain $\{ x_\alpha \}$ in $A$ and let $x = \bigvee x_\alpha$. Suppose $c \in \mathbb{K}L$ and $c \leq x$. There exists a finite subcollection from the chain such that $c \leq x_1 \vee \cdots \vee x_n \leq x$. Hence $c \leq x_i \leq x$, for some $i$. Since $x_i \in dL$ it follows that $c^{\perp\perp} \leq x_i \leq x$. Therefore $x \in dL$.
- If possible let $x = 1$, then $u \leq x$. As before, $u \leq x_j \in A$ for some $j$. Since $x_j \in dL$, $1 = u^{\perp\perp} \leq x_j$, contradicting the fact that $x_j < 1$. Hence $x < 1$, proving that $x \in A$.
- By Zorn’s Lemma, maximal $d$-elements exist in $L$.
- We use $\text{Max}(dL)$ to denote the (proper) maximal $d$-elements of $L$. 
Existence of $Max(dL)$:

- Suppose $L$ is an $M$-frame that contains a unit $u$.

- Let $A = \{ x \in dL : x < 1 \}$. Obviously $0 \in A$, which means that $A \neq \emptyset$.

- Take a chain $\{ x_\alpha \}$ in $A$ and let $x = \bigvee x_\alpha$. Suppose $c \in \kappa L$ and $c \leq x$. There exists a finite subcollection from the chain such that $c \leq x_1 \lor \cdots \lor x_n \leq x$. Hence $c \leq x_i \leq x$, for some $i$. Since $x_i \in dL$ it follows that $c \perp \perp \leq x_i \leq x$. Therefore $x \in dL$.

- If possible let $x = 1$, then $u \leq x$. As before, $u \leq x_j \in A$ for some $j$. Since $x_j \in dL$, $1 = u \perp \perp \leq x_j$, contradicting the fact that $x_j < 1$. Hence $x < 1$, proving that $x \in A$.

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- By Zorn’s Lemma, maximal $d$-elements exist in $L$.

- We use $\text{Max}(dL)$ to denote the (proper) maximal $d$-elements of $L$. 
Existence of $\text{Max}(d\mathbb{L})$:

- Suppose $L$ is an $M$-frame that contains a unit $u$.

- Let $A = \{ x \in d\mathbb{L} : x < 1 \}$. Obviously $0 \in A$, which means that $A \neq \emptyset$.

- Take a chain $\{ x_\alpha \}$ in $A$ and let $x = \bigvee x_\alpha$. Suppose $c \in \mathbb{L}$ and $c \leq x$. There exists a finite subcollection from the chain such that $c \leq x_1 \lor \cdots \lor x_n \leq x$. Hence $c \leq x_i \leq x$, for some $i$. Since $x_i \in d\mathbb{L}$ it follows that $c \bot \bot \leq x_i \leq x$. Therefore $x \in d\mathbb{L}$.

- If possible let $x = 1$, then $u \leq x$. As before, $u \leq x_j \in A$ for some $j$. Since $x_j \in d\mathbb{L}$, $1 = u \bot \bot \leq x_j$, contradicting the fact that $x_j < 1$. Hence $x < 1$, proving that $x \in A$.

- By Zorn’s Lemma, maximal $d$-elements exist in $L$.

- We use $\text{Max}(d\mathbb{L})$ to denote the (proper) maximal $d$-elements of $L$. 
Two important results:

**Theorem**

Let $x \in \text{Max}(dL)$, then $x$ is maximal with respect to $u \nleq x$ for all units $u$ of $L$. Conversely, if $y \in L$ is maximal with respect to $u \nleq x$ for all units $u$, then $y \in \text{Max}(dL)$.

**Theorem**

$\text{Min}(L) \subseteq dL$ and $\text{Max}(dL) \subseteq \text{Spec}(L)$. 
Two important results:

**Theorem**

Let \( x \in Max(dL) \), then \( x \) is maximal with respect to \( u \nless x \) for all units \( u \) of \( L \). Conversely, if \( y \in L \) is maximal with respect to \( u \nless x \) for all units \( u \), then \( y \in Max(dL) \).

**Theorem**

\( Min(L) \subseteq dL \) and \( Max(dL) \subseteq Spec(L) \).
We endow $\text{Max}(dL)$ with the subspace topology from $\text{Spec}(L)$, namely, the Zariski topology.

Define

$$U_d(x) = \{ m \in \text{Max}(dL) : x \not\succ m \}.$$ 

The following hold for all $c, k \in \mathbb{R}L$.

1. $U_d(c) = \text{Max}(dL)$ if and only if $c$ is a unit.
2. $\bigcup U_d(c) = U_d(\bigvee c)$.
3. $U_d(c) \cap U_d(k) = U_d(c \wedge k)$.
4. $U_d(c) = U_d(c^{\perp \perp})$.

The collection $\{ U_d(c) : c \in \mathbb{R}L \}$ forms a basis of open sets for the Zariski topology on $\text{Max}(dL)$. 
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$$U_d(x) = \{ m \in \text{Max}(dL) : x \nless m \}.$$ 

The following hold for all $c, k \in \mathcal{K}L$.

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2. $\bigcup U_d(c) = U_d(\bigvee c)$.
3. $U_d(c) \cap U_d(k) = U_d(c \land k)$.
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The collection \( \{ U_d(c) : c \in \mathbb{R} L \} \) forms a basis of open sets for the Zariski topology on \( \text{Max}(dL) \).
Theorem

Let $L$ be an $M$-frame that possesses a unit $u$. The space $\text{Max}(dL)$ is a compact topological space.

Important Observation: In case of a $W$-object $(G, u)$, $\text{Max}_d(G)$, w.r.t the Zariski topology is a compact, Hausdorff space. It so happens that to prove the Hausdorff condition, we require the property of "disjointification" which $\ell$-groups possess. This is the first result in frames that seems to NOT be occurring parallel to that of $\ell$-groups, since an algebraic frame does not satisfy disjointification always.

**Question:** If $L$ is an $M$-frame with a unit, when is $\text{Max}(dL)$ Hausdorff?
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Preliminaries

Primes and Minimal Primes

Max($dL$)

The space Max($dL$)

Ultrafilters of $\mathcal{K}L^\perp$

- It is known that

$$\text{Max}_d(G) \cong \text{Ult}(Z^\#(G)), \text{ for } W\text{-objects}$$

and

$$\text{Max}_d(X) \cong \text{Ult}(Z^\#(X)) \text{ for spaces}$$

where $Z^\#(X) = \{\text{cl int } Z : Z \in Z(X)\} \subseteq \mathcal{R}(X)$ (regular closed subsets of $X$)

- In frames, $\mathcal{B}(L) = \{x^\perp : x \in L\}$ is analogous to $\mathcal{R}(X)$ for spaces.

Q: In frames, what is analogous to $Z^\#(X)$?

- Let $\mathcal{R}L^\perp = \{c^\perp : c \in \mathcal{R}L\} \subseteq \mathcal{B}(L)$ be a subset.

**Lemma**

$\mathcal{R}L^\perp$ is a sublattice of $\mathcal{B}(L)$, with meet same as the meet of $L$ and the join is given by $x^\perp \lor y^\perp = (x \land y)^\perp$. Also, $\mathcal{R}L^\perp$ is a bounded lattice

- Filters and ultrafilters on $\mathcal{R}L^\perp$ exist.
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\[ \text{Max}_d(G) \cong \text{Ult}(Z^\#(G)), \text{ for W-objects} \]

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- Let \( \mathcal{K}L^\perp = \{ c^\perp : c \in \mathcal{K}L \} \subseteq \mathcal{B}(L) \) be a subset.

**Lemma**

\( \mathcal{K}L^\perp \) is a sublattice of \( \mathcal{B}(L) \), with meet same as the meet of \( L \) and the join is given by \( x^\perp \lor^\prime y^\perp = (x \land y)^\perp \). Also, \( \mathcal{K}L^\perp \) is a bounded lattice

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where \( Z^\#(X) = \{ \text{cl int } Z : Z \in Z(X) \} \subseteq R(X) \) (regular closed subsets of \( X \))

- In frames, \( B(L) = \{ x^\perp : x \in L \} \) is analogous to \( R(X) \) for spaces.

**Q:** In frames, what is analogous to \( Z^\#(X) \)?

- Let \( R(L) \perp = \{ c^\perp : c \in R(L) \} \subseteq B(L) \) be a subset.

**Lemma**

\( R(L) \perp \) is a sublattice of \( B(L) \), with meet same as the meet of \( L \) and the join is given by \( x^\perp \lor^\perp y^\perp = (x \land y)^\perp \). Also, \( R(L) \perp \) is a bounded lattice

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In frames, \( \mathcal{B}(L) = \{ x^\bot : x \in L \} \) is analogous to \( \mathcal{R}(X) \) for spaces.

**Q:** In frames, what is analogous to \( Z^\#(X) \)?

- Let \( \mathcal{R}L^\bot = \{ c^\bot : c \in \mathcal{R}L \} \subseteq \mathcal{B}(L) \) be a subset.

**Lemma**

\( \mathcal{R}L^\bot \) is a sublattice of \( \mathcal{B}(L) \), with meet same as the meet of \( L \) and the join is given by \( x^\bot \lor^\prime y^\bot = (x \wedge y)^\bot \). Also, \( \mathcal{R}L^\bot \) is a bounded lattice.

- Filters and ultrafilters on \( \mathcal{R}L^\bot \) exist.
It is known that
\[ \text{Max}_d(G) \cong \text{Ult}(Z^\#(G)), \text{ for } W\text{-objects} \]
and
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In frames, \( \mathcal{B}(L) = \{ x^\perp : x \in L \} \) is analogous to \( \mathcal{R}(X) \) for spaces.

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- Let \( \mathcal{K}L^\perp = \{ c^\perp : c \in \mathcal{K}L \} \subseteq \mathcal{B}(L) \) be a subset.

**Lemma**

\( \mathcal{K}L^\perp \) is a sublattice of \( \mathcal{B}(L) \), with meet same as the meet of \( L \) and the join is given by \( x^\perp \lor^\prime y^\perp = (x \land y)^\perp \). Also, \( \mathcal{K}L^\perp \) is a bounded lattice.

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- Filters and ultrafilters on $\mathcal{K}L^\perp$ exist.
Theorem

Suppose $L$ possesses a unit $u$.

1. If $F$ is a filter on $K\perp_L$, then $x(F) = \bigvee\{c\perp\perp : c\perp \in F\}$ is a proper $d$-element of $L$.

2. If $x \in dL$ is proper, then $\hat{F}_x = \{c\perp : c \in K_L, c \leq x\}$ is a filter of $K\perp_L$.

3. $\hat{F}_{x(G)} = G$, for every filter $G$ of $K\perp_L$ and $x(\hat{F}_y) = y$, for every $y \in dL$.

Let $\Phi_d : Ult(K\perp_L) \to \text{Max}(dL)$ be defined by $\Phi_d(U) = x(U)$.

Theorem

Let $L$ be an $M$-frame with a unit. $\Phi_d$ is a well-defined bijection with $\Phi_d^{-1}(m) = \hat{F}_m$. 

**Theorem**

Suppose $L$ possesses a unit $u$.

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Suppose $L$ possesses a unit $u$.

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Theorem

Let $L$ be an $M$-frame with a unit. $\Phi_d$ is a well-defined bijection with $\Phi_d^{-1}(m) = \hat{F}_m$. 
Next, we topologize $\text{Ult}(\mathcal{K} \mathcal{L})$ with the well-known Wallman topology: basic open sets are $M(l^\perp) = \{ U \in \text{Ult}(\mathcal{K} \mathcal{L}) : l^\perp \notin U \}$, for $l \in \mathcal{K} \mathcal{L}$.

Finally, the main result:

**Theorem**

Let $L$ be an $M$-frame with a unit $u$. The map $\Phi_d : \text{Ult}(\mathcal{K} \mathcal{L}) \to \text{Max}(dL)$ is a homeomorphism between the topological spaces $\text{Ult}(\mathcal{K} \mathcal{L})$, with respect to the Wallman topology, and $\text{Max}(dL)$, endowed with the hull-kernel topology.

So, $\text{Max}(dL) \cong \text{Ult}(\mathcal{K} \mathcal{L})$
Next, we topologize $\text{Ult}(\mathbb{K}L^\perp)$ with the well-known Wallman topology: basic open sets are $M(l^\perp) = \{ U \in \text{Ult}(\mathbb{K}L^\perp) : l^\perp \notin U \}$, for $l \in \mathbb{K}L$.

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So, $\text{Max}(dL) \cong \text{Ult}(\mathbb{K}L^\perp)$
Preliminaries
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The space $\text{Max}(dL)$
Ultrafilters of $L^\perp$

- If $U \in \text{Ult}(L)$, then $U^* = \{ c^\perp : c \in L \setminus U \}$ is a prime filter of $L^\perp$.
- If $V \in \text{Ult}(L^\perp)$, then $V^* = \{ c \in L : c^\perp \in V \}$ is a prime filter of $L$.

Q: When can we have a well-defined bijection between $\text{Ult}(L)$ and $\text{Ult}(L^\perp)$?

Answer: Complemented frames

Theorem

The following are equivalent for an M-frame $L$ that possesses a unit $u$.

1. $L$ is a complemented frame.
2. $\text{Min}(L) = \text{Max}(dL)$.
3. $\text{Min}(L)$ is homeomorphic to $\text{Max}(dL)$.
4. $\text{Ult}(L)^{-1} \cong \text{Ult}(L^\perp)$, a homeomorphism.
If $U \in \text{Ult}(\mathfrak{L})$, then $U^* = \{c^\perp : c \in \mathfrak{L} \setminus U\}$ is a prime filter of $\mathfrak{L}^\perp$.

If $V \in \text{Ult}(\mathfrak{L}^\perp)$, then $V^*_\perp = \{c \in \mathfrak{L} : c^\perp \in V\}$ is a prime filter of $\mathfrak{L}$.

Q: When can we have a well-defined bijection between $\text{Ult}(\mathfrak{L})$ and $\text{Ult}(\mathfrak{L}^\perp)$?

Answer: Complemented frames

**Theorem**

The following are equivalent for an $M$-frame $L$ that possesses a unit $u$.

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We know,

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