Minor Identities and Primitive Positive Constructions

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Outline

- Primitive positive constructions

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- Minor-preserving maps

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- Infinite structures

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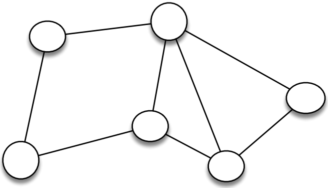
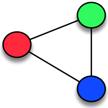
CSP$(\underline{B})$: 3-colorability problem.





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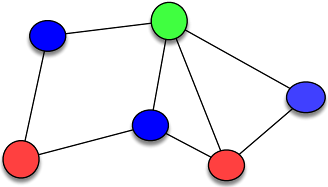

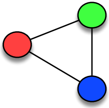

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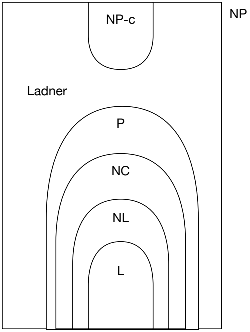

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Open questions:

- When is CSP($\underline{B}$) in NC?
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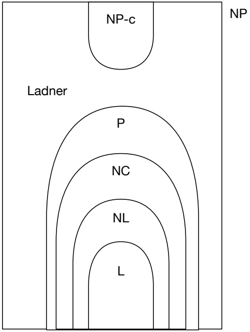

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- Study polymorphism clones over finite sets with respect to minor-preserving maps.

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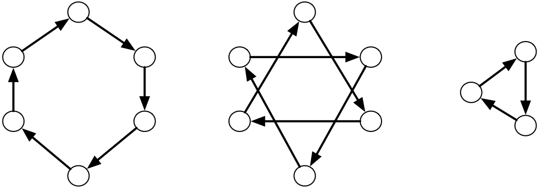

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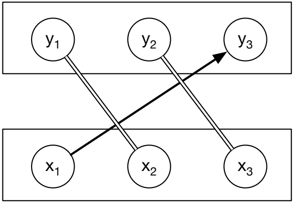

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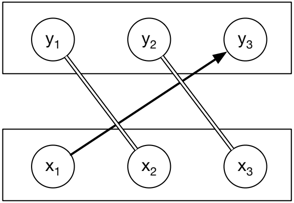

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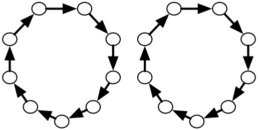

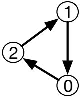

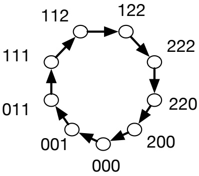

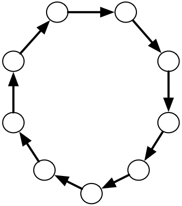

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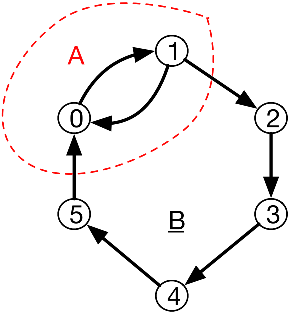

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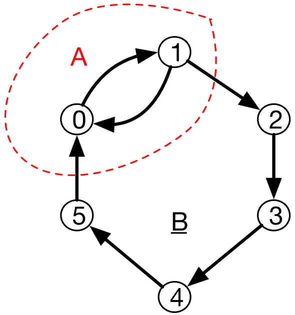

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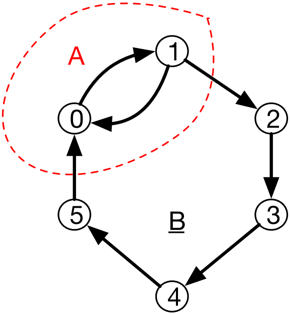

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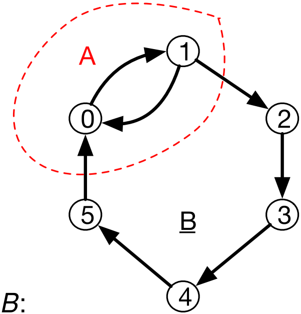

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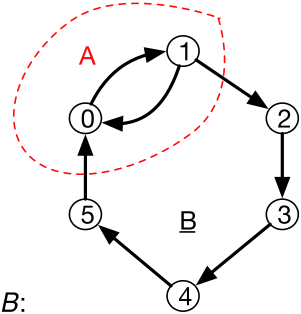

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Fact for finite B:
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A is primitively positively definable in $\underline{B}$.

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$$\big((a_1^1,\dots,a_1^d),\dots,(a_k^1,\dots,a_k^d)\big) \mapsto \big(f(a_1^1,\dots,a_k^1),\dots,f(a_1^d,\dots,a_k^d)\big)$$

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$$\big((a_1^1,\dots,a_1^d),\dots,(a_k^1,\dots,a_k^d)\big) \mapsto \big(f(a_1^1,\dots,a_k^1),\dots,f(a_1^d,\dots,a_k^d)\big)$$

for each $f \in \text{Pol}(\underline{A})$.

Fact.

$$\text{Pol}(\underline{A})^d = \text{Pol}(\underline{A}^d; E_{1,2},\dots,E_{d-1,d})$$

where $E_{i,j} := \{(s,t) \in A^d \mid s_i = t_j\}$.

PP-Definitions and Homomorphic Images

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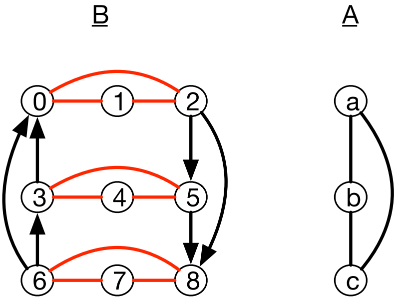

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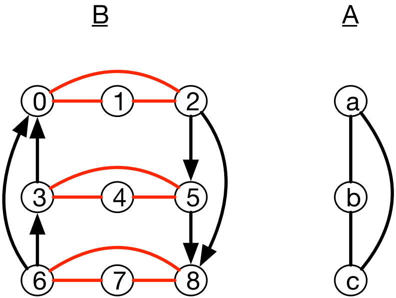

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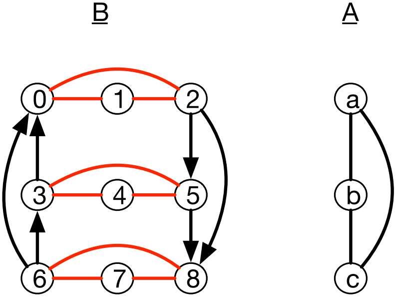

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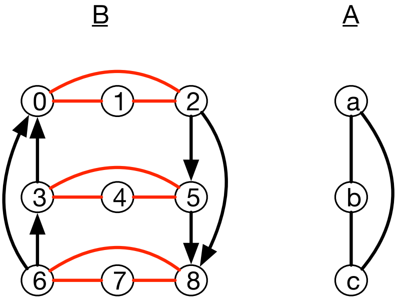

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Example: $h \colon B \to A$ with

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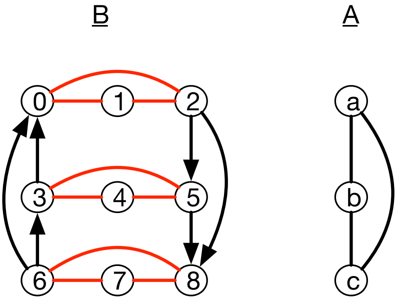

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- $h^{-1}(=)$ has pp definition $\psi(x, y) := \exists u(E(x, u) \wedge E(u, y))$
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PP-Interpretations and Clone Homomorphisms

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\text{Primitive positive interpretations} \arrow[dr, leftrightarrow] & & & \text{Clone homomorphisms} \\
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How about this connection for primitive positive constructions?

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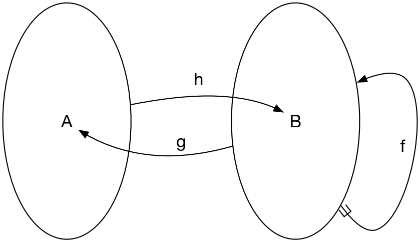

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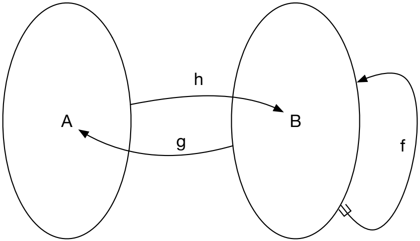

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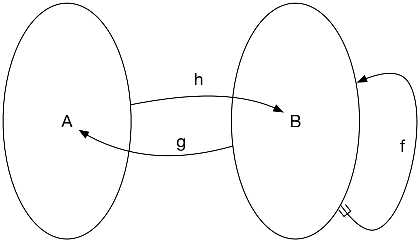

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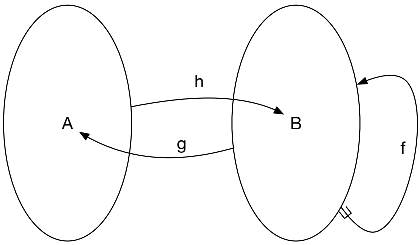

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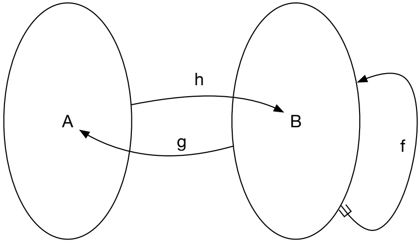

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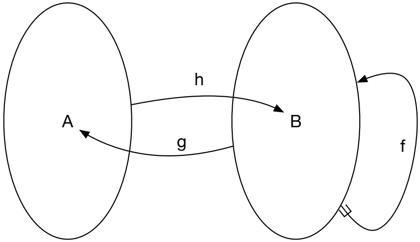

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Note. $\text{Exp}\big(\text{Refl}(\text{P}^{\text{fin}}(\text{Pol}(\underline{B})))\big)$ contains $\text{HSP}^{\text{fin}}(\text{Pol}(\underline{B}))$.

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$\mathcal{C}$: set of operations.
Write $\mathcal{C} \models \Sigma$ if Σ is satisfied by some operations in $\mathrm{Pol}(\underline{B})$
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Minor Conditions and Reflections

Σ: minor condition.
If $\text{Pol}(\underline{B}) \models \Sigma$ and $\xi\colon \text{Pol}(\underline{B}) \to \text{Pol}(\underline{A})$ is minor-preserving,
then $\text{Pol}(\underline{B}) \models \Sigma$.

Theorem (Barto+Opršal+Pinsker).

Let $\underline{A}$ be finite. And let $\underline{B}$ be finite, too. Then TFAE:

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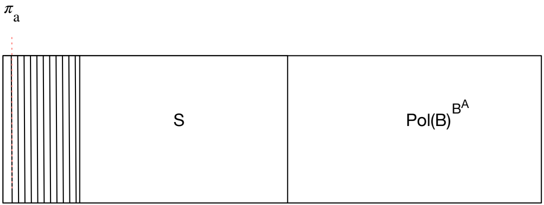

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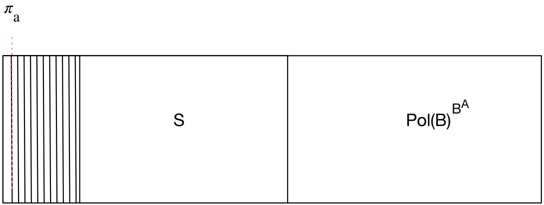

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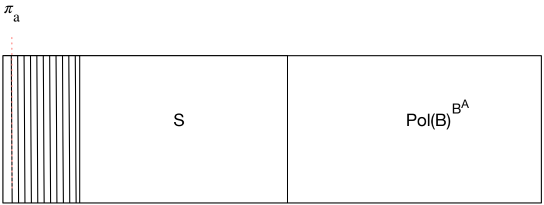

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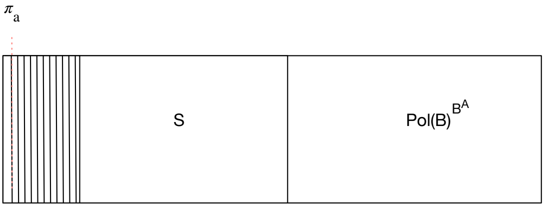

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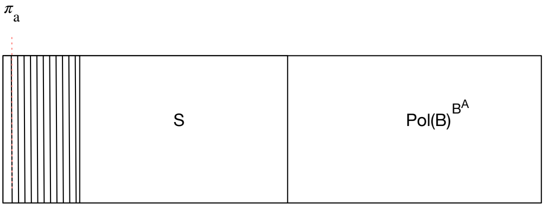

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Reflection pseudo-varieties $\longleftrightarrow$ Primitive positive constructions

Height one identities $\longleftrightarrow$ Minor-preserving maps

Primitive positive interpretations $\longleftrightarrow$ Pseudo-varieties

Clone homomorphisms $\longleftrightarrow$ Identities

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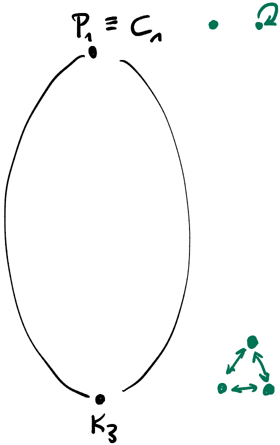

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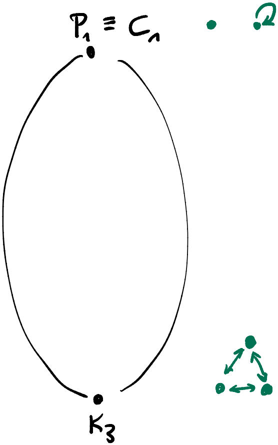

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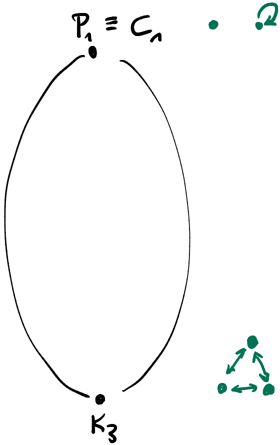

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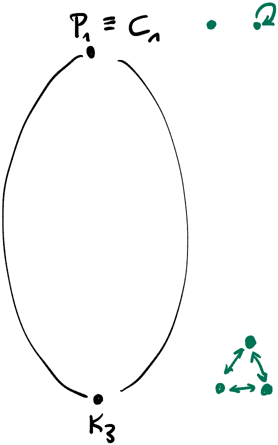

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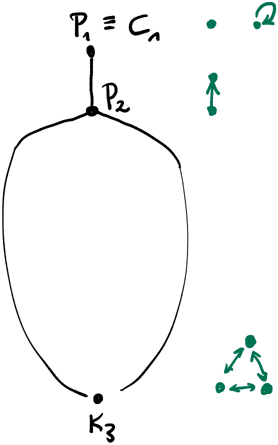

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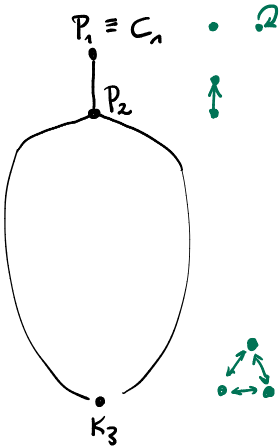

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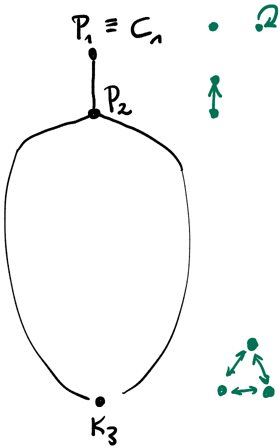

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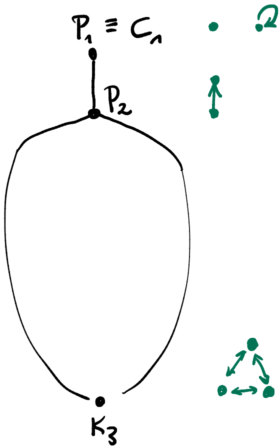

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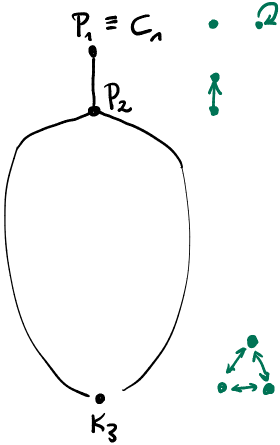

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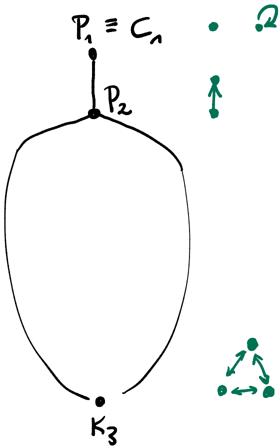

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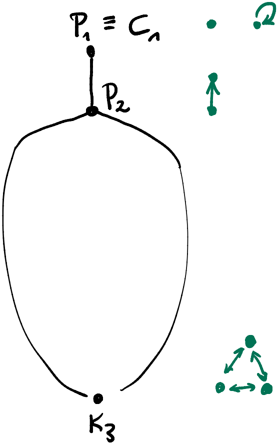

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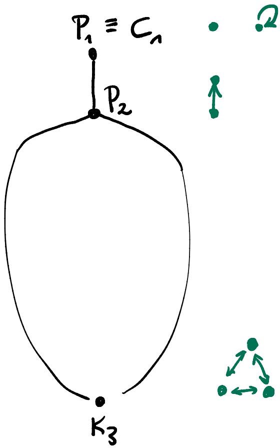

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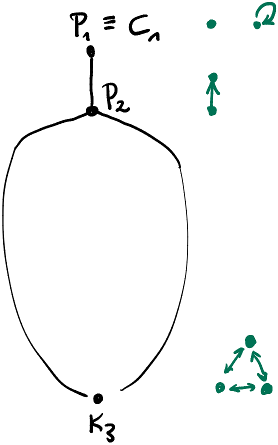

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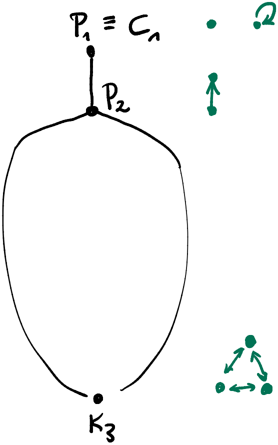

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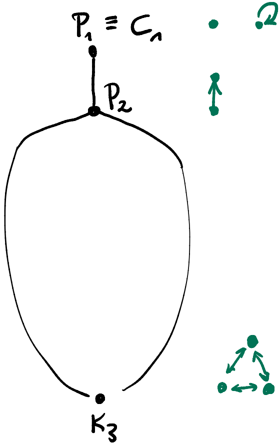

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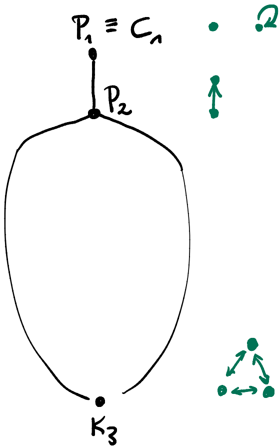

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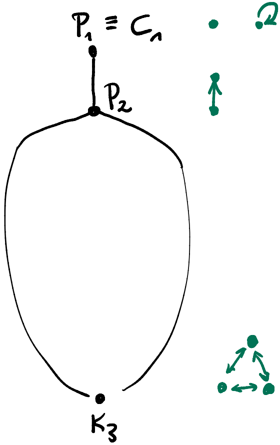

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Consequence: Can focus on idempotent clones $\mathcal{C}$,
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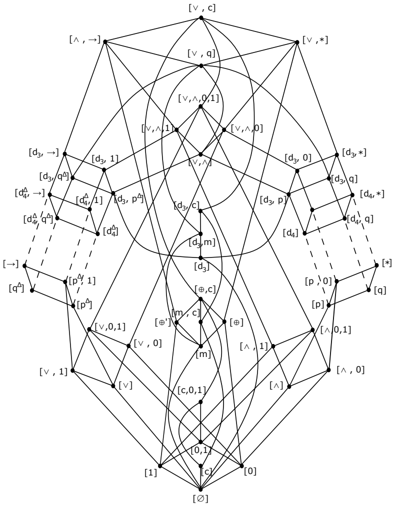
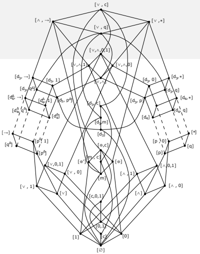


Post's lattice

The constructability poset on $\{0, 1\}$



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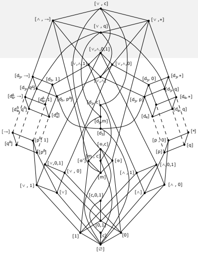
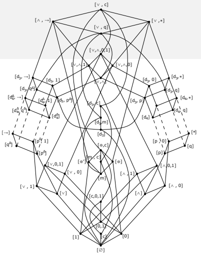


Pieces:

The constructability poset on $\{0, 1\}$



Pieces:

 $\mathcal{C}$ collapses with $\mathcal{C}^* := \{\neg f(\neg x_1, \ldots, \neg x_n) \mid f \in \mathcal{C}\}$.

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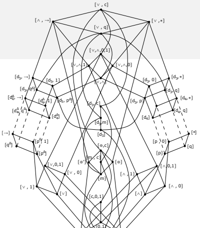

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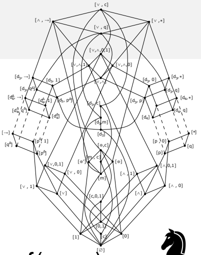

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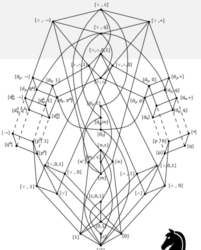

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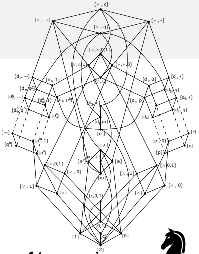

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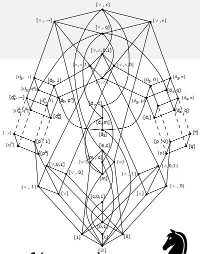
 $(\{0,1\}; \leq, \{0\}, \{1\}, \{(0,1),(1,0),(1,1)\})$
and $(\{0,1\}; \leq, \{0\}, \{1\})$ collapse (2D)

The constructability poset on $\{0, 1\}$



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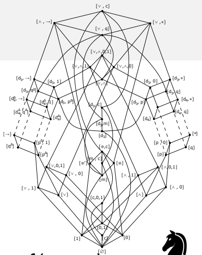
Jonsson(3) 

The constructability poset on $\{0,1\}$



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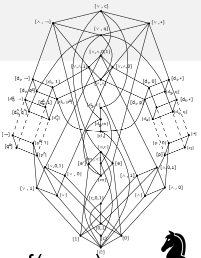
Hagemann-Mitschke(3) 

The constructability poset on $\{0, 1\}$



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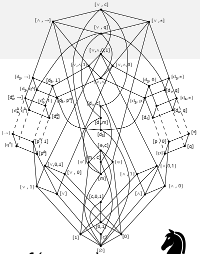
Hagemann-Mitschke(3)

QNU(4)

The constructability poset on $\{0, 1\}$



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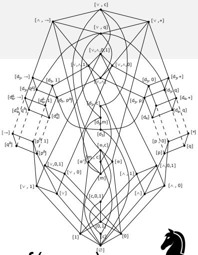
Jonsson(3) 

Hagemann-Mitschke(3) 

QNU(4) 

QNU(5) 

The constructability poset on $\{0,1\}$



Pieces:

♘ $\mathcal{C}$ collapses with $\mathcal{C}^* := \{\neg f(\neg x_1, \ldots, \neg x_n) \mid f \in \mathcal{C}\}$.

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f(x,y) = f(y,x) ♜

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Jonsson(3) ♛

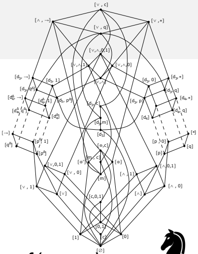
Hagemann-Mitschke(3) ♚

QNU(4) ♟

QNU(5) ♟

QNU(6) ♟

The constructability poset on $\{0,1\}$



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Hagemann-Mitschke(3) 

QNU(4) 

QNU(5) 

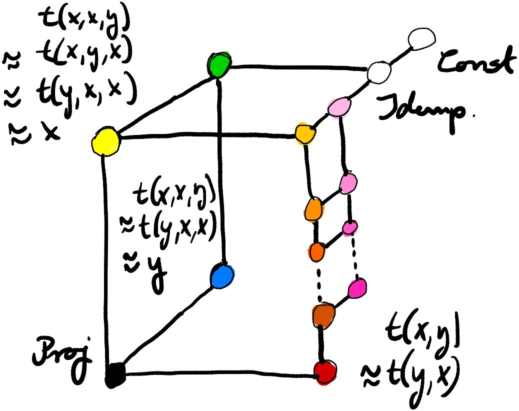
QNU(6) 

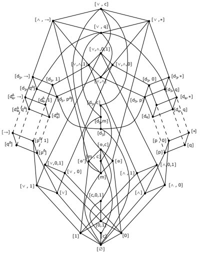
...

The Constructability Poset

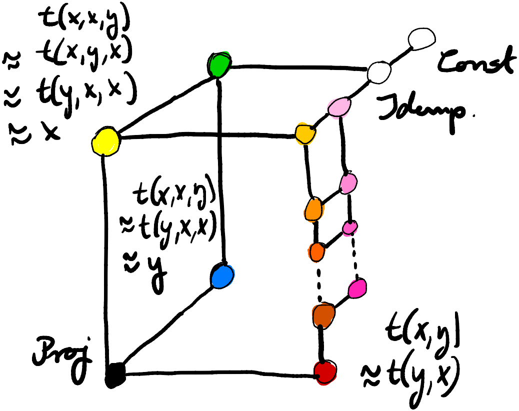
$\leq_{\text{con}}$ on $\{0,1\}$: outcome (B. + Vucaj 2020)

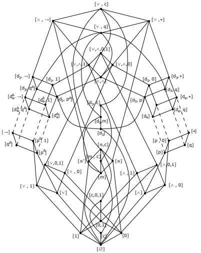
The Constructability Poset





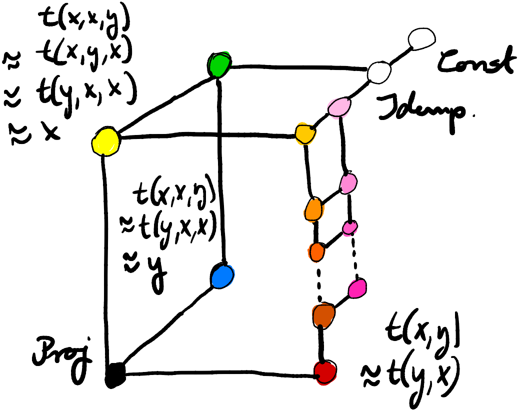
The Constructability Poset

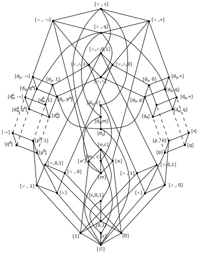




Strictly above black: CSP in P (Schaefer'78).

The Constructability Poset





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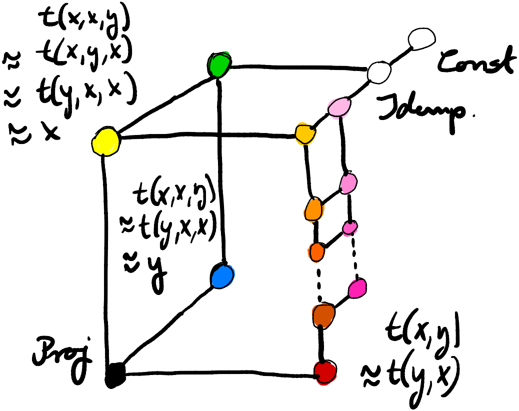
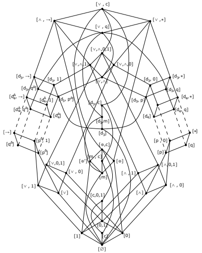
Above yellow or red iff CSP in Datalog.

The Constructability Poset





Above blue or yellow:
CSP is in NC.

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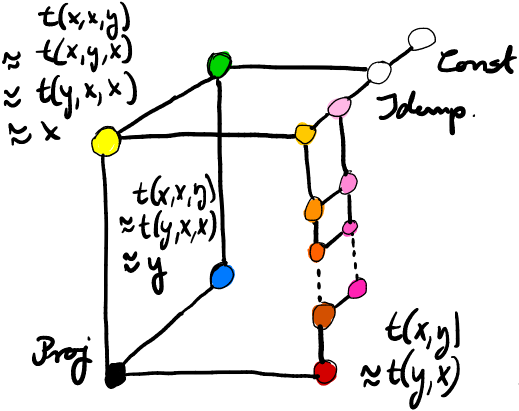
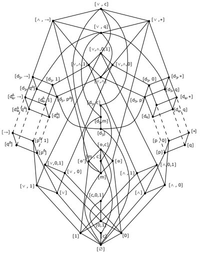
The Constructability Poset





Above blue or yellow:
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Above yellow or deep orange:
CSP in NL.

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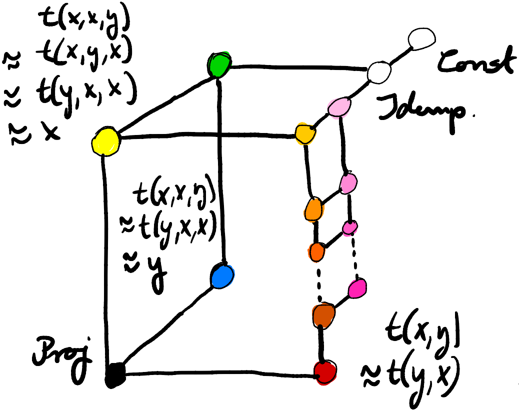

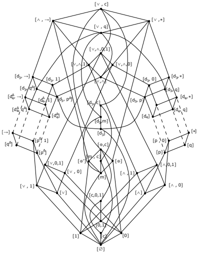

Above blue or yellow:
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CSP in NL.
Above yellow or deep purple:
CSP in L.

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Clones over three elements

$\leq_{\text{def}}$ on $\{0, 1, 2\}$:

Clones over three elements

$\leq_{\text{def}}$ on $\{0, 1, 2\}$:

Yanov-Muchnik: 2^ω

Clones over three elements

$\leq_{\text{def}}$ on $\{0,1,2\}$:

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How about $\leq_{\text{int}}$?

The interpretability poset on $\{0, 1, 2\}$

$\leq_{\text{int}}$ on $\{0, 1, 2\}$: 

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$$C_3 := \{(0,1),(1,2),(2,0)\}$$
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$$R_3^{=} := \{(x,y,z) \mid x \in \{0,1\} \wedge x = 0 \Rightarrow y = z\}$$

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Zhuk'15: 2^ω many clones between

$$\mathrm{Pol}(\{0,1,2\}; C_3, R_3^{=})$$
$$\text{and} \quad \mathrm{Pol}(\{0,1,2\}; C_3, B_2)$$

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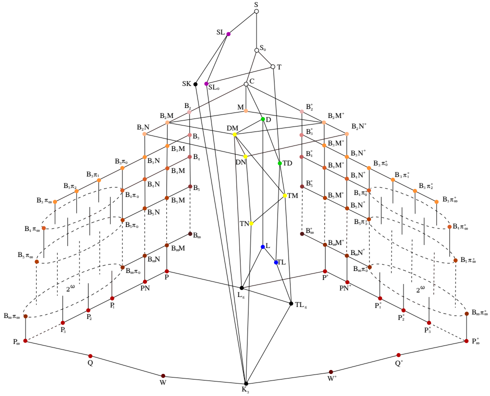
$$\begin{array}{rl} & \mathrm{Pol}(\{0,1,2\}; C_3, R_3^{=}) \\ \text{and} & \mathrm{Pol}(\{0,1,2\}; C_3, B_2) \end{array}$$

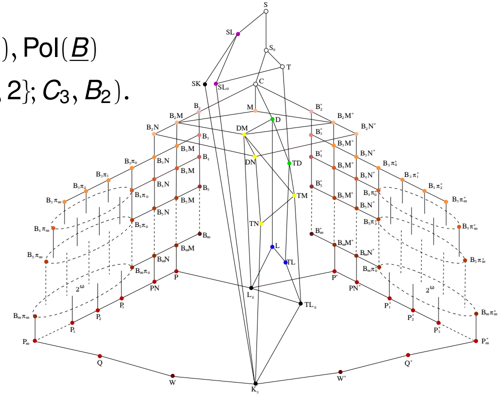
- Clones below $\mathrm{Pol}(\{0,1,2\}; C_3)$: 'self-dual'
- $\mathrm{Pol}(\{0,1,2\}; C_3, R_3^{=})$ contains binary 'paper-scissor-stone operation'

Non-collapse



Theorem (Zhuk). Let $\underline{A}$ and $\underline{B}$ be structures s.t.

$$(\{0,1,2\}; C_3, R_3^{=}) \leq_{\text{def}} \text{Pol}(\underline{A}), \text{Pol}(\underline{B})$$
$$\leq_{\text{def}} (\{0,1,2\}; C_3, B_2).$$



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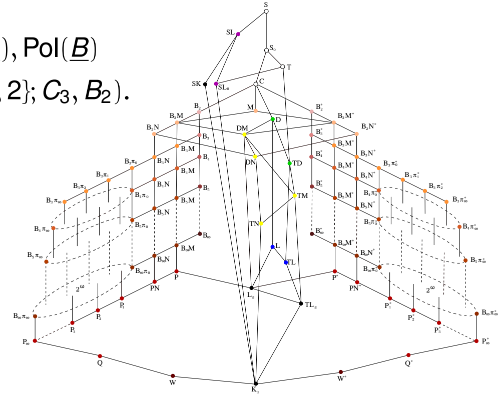
If $\underline{B} \leq_{\text{int}} \underline{A}$ then $\underline{B} \leq_{\text{def}} \underline{A}$.



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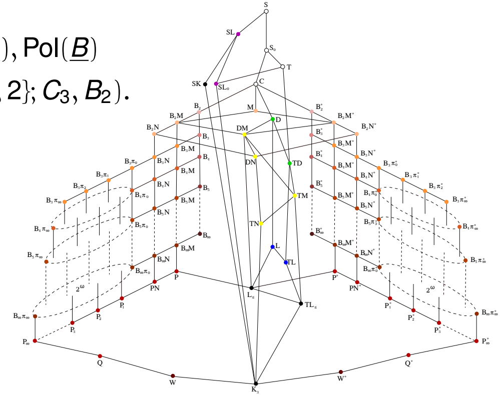
If $\underline{B} \leq_{\text{int}} \underline{A}$ then $\underline{B} \leq_{\text{def}} \underline{A}$.

Corollary: 2^ω clones over $\{0,1,2\}$ even when considered up to clone homomorphism equivalence!



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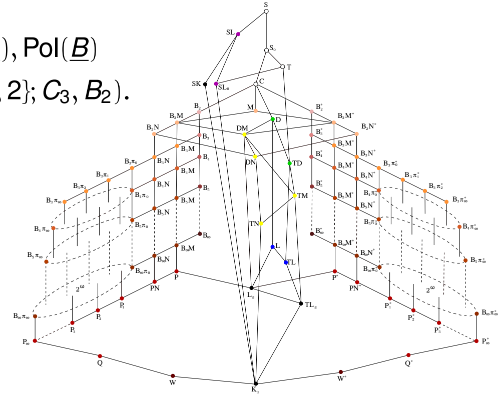

Conclusion: Need stronger weapon.

Non-collapse

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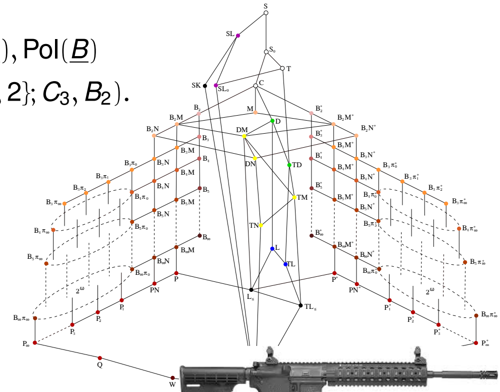

Conclusion: Need stronger weapon.

pp constructions!

The constructability poset on $\{0, 1, 2\}$

The constructability poset on $\{0, 1, 2\}$

$W := (\{0, 1, 2\}; C_3, R_3^{=})$

The constructability poset on $\{0, 1, 2\}$

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3-4 weak near unanimity

$$g(x, x, y) = g(x, y, x) = g(y, x, x) =$$
$$f(x, x, x, y) = f(x, x, y, x) = f(x, y, x, x) = f(y, x, x, x)$$

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'guarded 3-cyclic':
$f(x, x, x, y) = x, f(x_1, x_2, x_3, y) = f(x_2, x_3, x_1, y)$

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 $Q \equiv P := (\{0,1,2\}; C_3, \{(x,y,z) \mid x,y \in \{0,1\} \wedge x = y = 0 \Rightarrow z = 0)$

(Infinitely many clones between Q and P.)

The constructability poset on $\{0, 1, 2\}$

$W := (\{0, 1, 2\}; C_3, R_3^{=})$

3-4 weak near unanimity

$$g(x, x, y) = g(x, y, x) = g(y, x, x) = f(x, x, x, y) = f(x, x, y, x) = f(x, y, x, x) = f(y, x, x, x)$$

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 Further collapses ...

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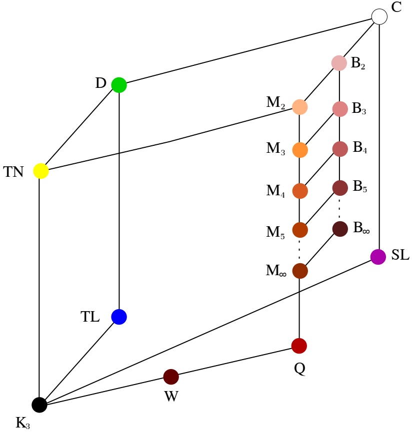

Digraphs

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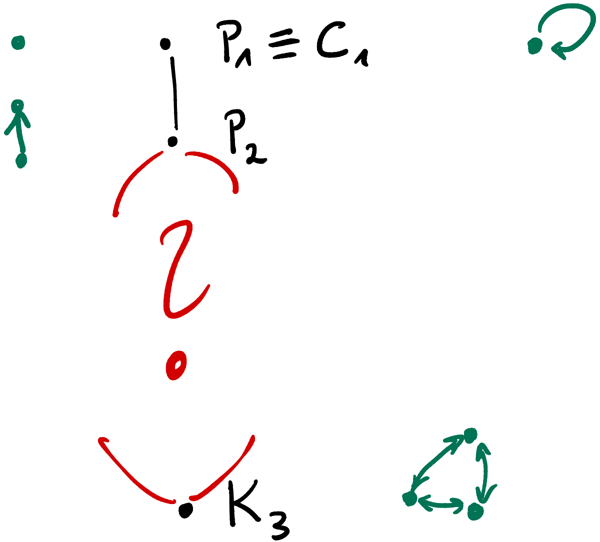

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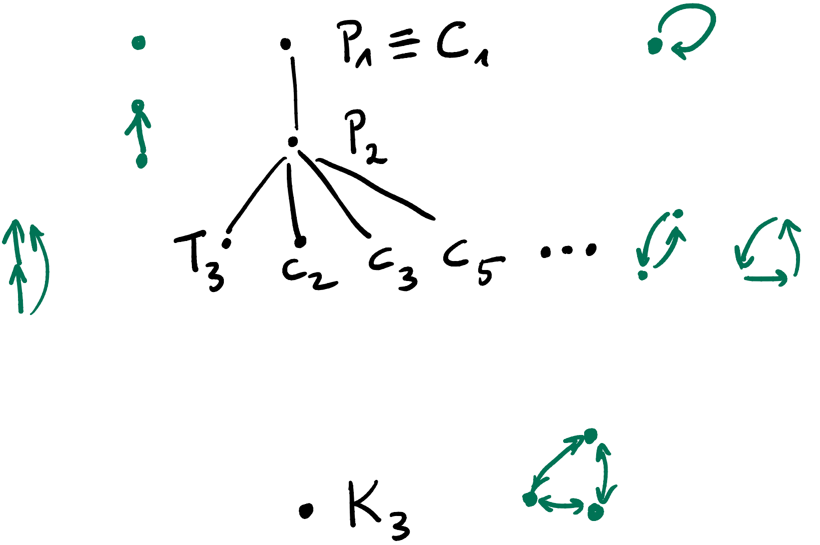

Digraphs: pieces

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$P_2 \leq_{\text{con}} D$ for every digraph D
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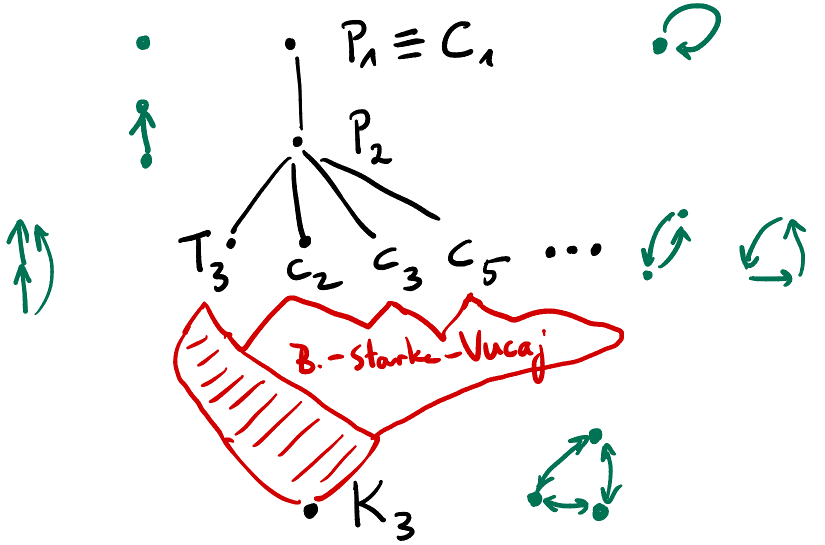
Joint work with F. Starke

Digraphs: current state



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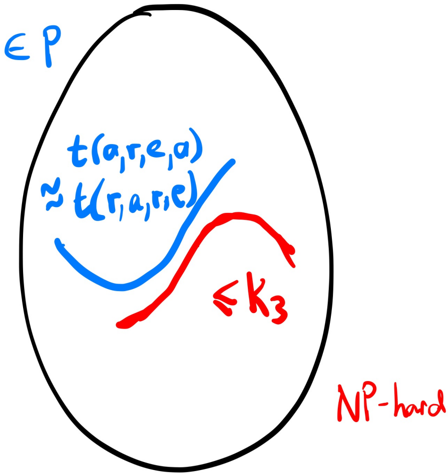
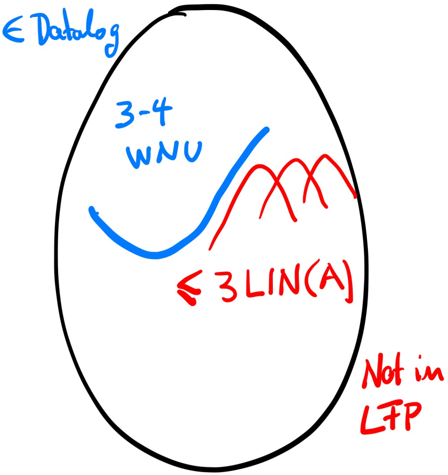
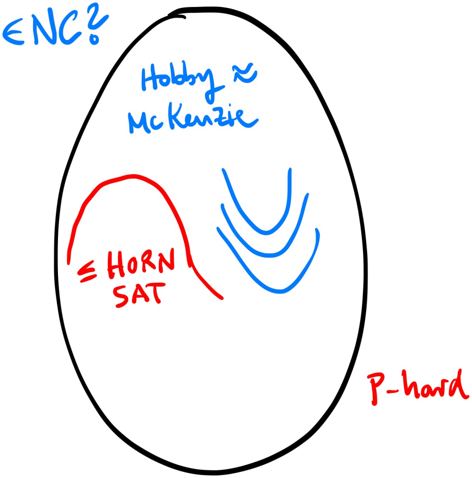
Algebraic Dichotomy, Pictorially



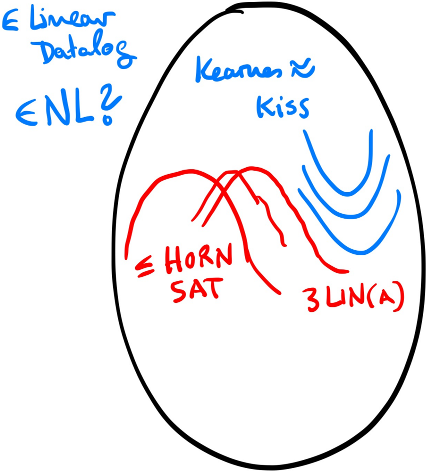
The Barto+Kozik Theorem



PP Obstruction Theorem 3



PP Obstruction Theorem 4



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Conjecture (B.+Pinsker'11).

Let $\underline{B}$ be a reduct of a finitely bounded homogeneous structure.
If $\underline{B} \not\leq_{\text{con}} K_3$ then CSP($\underline{B}$) is in P.