States over Płonka sums of Boolean algebras

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(joint work with A. Loi)

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IIIA - CSIC

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Structure of the talk

1. Preliminaries
2. States on Płonka sums of Boolean algebras
3. Faithful states
4. Induced topology

Motivation

In probability theory, events are assumed to be formulas of classical logic (elements of a Boolean algebra).

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2. fuzzy events
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1. Quantum events — states on orthomodular lattices (Bennet, Foulis, Greechie)
2. fuzzy events — states on MV-algebras, Goedel, product algebras, ... (Mundici, Flaminio, Kroupa, Aguzzoli, Marra, Gerla, Godo, Ugolini, ...)
3. three-valued events — states on De Morgan algebras (Negri).

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Semilattice direct systems

Definition (direct system)

A semilattice direct system of algebras is

$$\mathbb{A} = \langle \{\mathbf{A}_i\}_{i \in I}, (p_{ij} : i \leqslant j), \mathbf{I} \rangle$$

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- $\mathbf{I} = \langle I, \leqslant \rangle$ is a semilattice with least element i_0;
- for each $i \leqslant j$,there exists a homomorphism $p_{ij} : \mathbf{A}_i \to \mathbf{A}_j$ s.t.
  1. p_{ii} is the identity in $\mathbf{A}_i$
  2. for every $i \leq j \leq k$, $p_{jk} \circ p_{ik} = p_{ik}$

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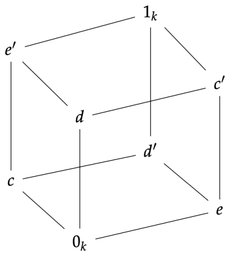
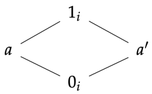
  $$g^{\mathcal{P}_l(\mathbf{A}_i)}(a_1, \dots, a_n) := g^{\mathbf{A}_j}(p_{i_1 j}(a_1), \dots, p_{i_n j}(a_n)).$$
- if $g \in \tau$ is a constant, then $g^{\mathcal{P}_l(\mathbf{A}_i)} = g^{\mathbf{A}_{i_0}}$.

Example

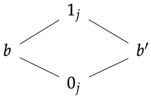
$\mathbf{A}_k =$



$\mathbf{A}_i =$



$\mathbf{A}_j =$



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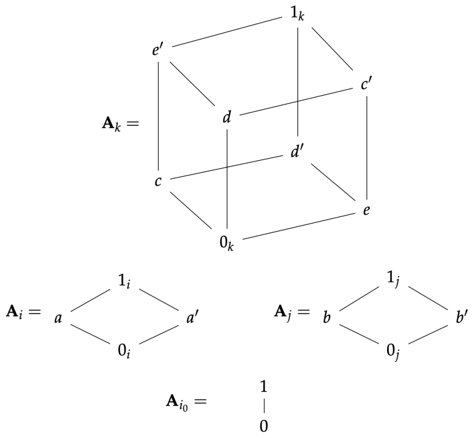

- i_0 least element (index set), $i, j \leq k$.

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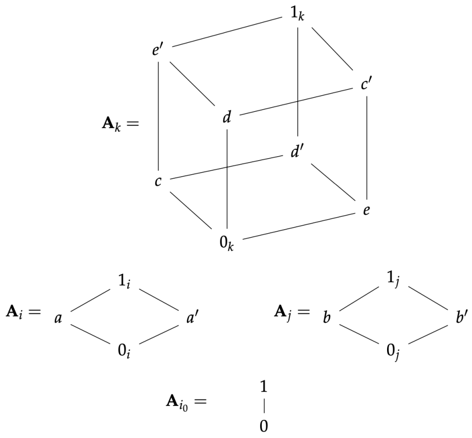

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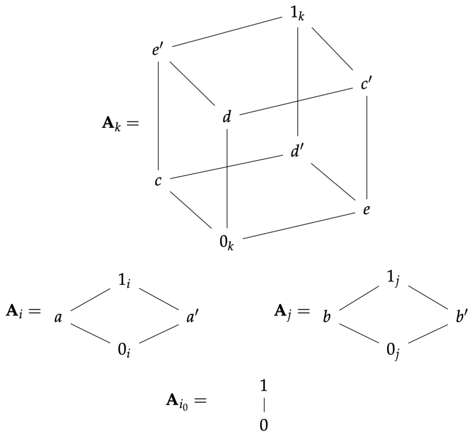

- i_0 least element (index set), $i, j \leq k$.
- $p_{ik}(a) = c$, $p_{jk}(b) = e$.
- $a \wedge^{\mathbf{B}} b = p_{ik}(a) \wedge^{\mathbf{A}_k} p_{jk}(b) = c \wedge^{\mathbf{A}_k} e = 0_k$.

Involutive bisemilattices

Definition

An involutive bisemilattice is an algebra $\mathbf{B} = \langle B, \wedge, \vee, ', 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ satisfying:

1. $x \vee x \approx x$;
2. $x \vee y \approx y \vee x$;
3. $x \vee (y \vee z) \approx (x \vee y) \vee z$;
4. $(x')' \approx x$;
5. $x \wedge y \approx (x' \vee y')'$;
6. $x \wedge (x' \vee y) \approx x \wedge y$;
7. $0 \vee x \approx x$;
8. $1 \approx 0'$.

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The class of involutive bisemilattices forms a variety which we denote by $\mathcal{IBSL}$.

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- Every involutive bisemilattice is isomorphic to a Płonka sum (over a semilattice direct system) of Boolean algebras.

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  Notation: 1_i, 0_i the top and bottom element, respectively, of the Boolean algebras $\mathbf{A}_i$ (in $\mathcal{P}_l(\mathbf{A}_i)$).

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- $\mathcal{IBSL}$ (and one of its subquasivariety) is the algebraic counterpart of weak Kleene logics (Bochvar and Paraconsistent weak Kleene).

SB, J. Gil-Férez, F. Paoli and L. Peruzzi.
On Paraconsistent Weak Kleene Logic: axiomatization and algebraic analysis.
Studia Logica, 105(2):253–297, 2017.

SB and M. Pra Baldi.
Containment logics: algebraic counterparts and reduced models.
Submitted, 2021.

Probability measure for Boolean algebras

Definition

Let $\mathbf{A}$ be a Boolean algebra. A finitely additive probability measure on $\mathbf{A}$ is a real-valued map $m\colon \mathbf{A} \to [0,1]$ such that:

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- $\mathcal{S}(\mathbf{A})$: the space of all the (finitely additive) probability measures over a Boolean algebra $\mathbf{A}$.
- every $a \in A$ can be uniquely associated to a function $a^*\colon \mathbf{A}^* \to [0,1]$, with $\mathbf{A}^*$ the dual Stone space of $\mathbf{A}$. (Belluce-Chang Theorem for semi-simple MV-algebras).

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Theorem (Kroupa-Panti)

Let $\mathbf{A}$ *be a Boolean algebra and* $m\colon \mathbf{A} \to [0,1]$ *a (finitely additive) probability measure. Then*

1. *There is a homeomorphism* $\Psi\colon \mathcal{S}(\mathbf{A}) \to \mathcal{M}(\mathbf{A}^*)$*, where* $\mathcal{M}(\mathbf{A}^*)$ *is the space of all regular Borel probability measures on its dual space* $\mathbf{A}^*$*.*

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1. *There is a homeomorphism $\Psi\colon \mathcal{S}(\mathbf{A}) \to \mathcal{M}(\mathbf{A}^*)$, where $\mathcal{M}(\mathbf{A}^*)$ is the space of all regular Borel probability measures on its dual space $\mathbf{A}^*$.*
2. *For every $a \in A$,*

$$m(a) = \int_{\mathbf{A}^*} a^*(M)\, d\mu_s(M),$$

with a^* is the unique function associated to a, $M \in \mathbf{A}^*$ and $d\mu_m = \Psi(m)$.

Booleanisation

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Direct limit

The direct limit on $\mathbb{A}$ is the Boolean algebra defined as the quotient:

$$\lim_{\to i \in I} \mathbf{A}_i := (\bigcup_{i \in I} A_i)/\sim,$$

where, for $a \in A_i$ and $b \in A_j$, $a \sim b$ if and only if there exist $c \in A_k$, for some $k \in I$ with $i, j \leq k$ such that $p_{ik}(a) = c = p_{jk}(b)$.

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- We call the direct limit $\mathbf{A}_\infty$ the Booleanisation of $\mathbf{B}$.

States over involutive bisemilattices

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Let $\mathbf{B}$ be an involutive bisemilattice. A state over $\mathbf{B}$ is a map $s\colon \mathbf{B} \to [0,1]$ such that:

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Condition 2 renders "logical" incompatibility of two elements.

Basic properties

Proposition

Let s be a state over an involutive bisemilattice $\mathbf{B}$. Then

1. $s(0) = 0$;
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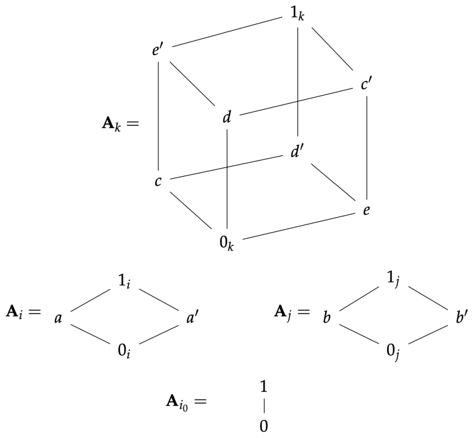
Proposition

Let s be a map from $\mathbf{B}$ to $[0,1]$. The following are equivalent:

1. s *is a state over* $\mathbf{B}$;
2. $s_i \colon \mathbf{A}_i \to [0,1]$ *is a (finitely additive) probability measure over* $\mathbf{A}_i$, *for every* $i \in I$, *and* p_{ij} *preserves the measures for each* $i \leq j$ *(*$s_j \circ p_{ij} = s_i$*).*

Example



- $s(a) = s(a') = \frac{1}{2}$,
- $s(b) = \frac{1}{3},\ s(b') = \frac{2}{3}$,
- $s(c) = s(c') = \frac{1}{2},\ s(d) = \frac{1}{6},\ s(e) = \frac{1}{3},\ s(d') = \frac{5}{6},\ s(e') = \frac{2}{3}$.

Existence of states

Theorem

Let $\mathbf{B} \in \mathcal{IBSL}$ with $\mathcal{P}_l(\mathbf{A}_i)$ its Płonka sum representation. The following are equivalent:

1. $\mathbf{B}$ *carries a state;*
2. $\mathcal{P}_l(\mathbf{A}_i)$ *contains no trivial algebra.*

States and Booleanisation

$\mathbf{B} \in \mathcal{IBSL}$, $\mathbf{A}_\infty$ the Booleanisation, $\pi\colon \mathbf{B} \to \mathbf{A}_\infty$,

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There exists a bijection $\Phi\colon \mathcal{S}(\mathbf{B}) \to \mathcal{S}(\mathbf{A}_\infty)$ such that Φ is state preserving, i.e. $s(b) = \Phi(s)(\pi(b))$.

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- Integral representation of states (above theorem + Kroupa-Panti).

Faithful states

Proposition

Let s be a state over $\mathbf{B} \in \mathcal{IBSL}$. The following are equivalent:

1. *s is faithful;*
2. *$s_i \colon \mathbf{A}_i \to [0,1]$ is a regular (finitely additive) probability measure over $\mathbf{A}_i$, for every $i \in I$, and p_{ij} is an injective homomorphism preserving the measures, for each $i \leq j$.*

Faithful states and Booleanisation

$\mathbf{B} \in \mathcal{IBSL}$ is injective if, for every $i, j \in I$ such that $i \leq j$, the homomorphism $p_{ij} \colon \mathbf{A}_i \to \mathbf{A}_j$ is injective.

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Theorem

Let $\mathbf{B} \in \mathcal{IBSL}$ injective. Then there is a bijective correspondence between faithful states over $\mathbf{B}$ and regular measures over $\mathbf{A}_\infty$.

Faithful states and Booleanisation

$\mathbf{B} \in \mathcal{IBSL}$ is injective if, for every $i, j \in I$ such that $i \leq j$, the homomorphism $p_{ij} \colon \mathbf{A}_i \to \mathbf{A}_j$ is injective.

Theorem

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- The class of *injective* $\mathcal{IBSL}$ is the algebraic counterpart of an extension of the logic PWK.

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Pseudometric

$\mathbf{B} \in \mathcal{IBSL}$ carrying a state s. Consider the term function $\triangle \colon B \times B \to B$, $a \triangle b \coloneqq (a \wedge b') \vee (a' \wedge b)$, for $a, b \in B$.

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- The map $d_s \colon \mathbf{B} \to [0, 1]$, $d_s := s \circ \triangle$ is a pseudo-metric on B, i.e.

1. $d_s(a, b) \geqslant 0$,
2. $d_s(a, a) = 0$,
3. $d_s(a, b) = d(b, a)$,
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for all $a, b, c \in B$.

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- A pseudo-metric d is a metric in case $d(x,y) = 0$ iff $x = y$.

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$U \subset B$ is open if and only if for every $x \in U$, there exists $r > 0$ s.t. $D_r(x) = \{y \in B | d_s(x, y) < r\} \subset U$.

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- d_∞ is a metric, d_s is not.

Sections

- $f\colon X \to Y$ surjective map between topological spaces X and Y, a *section* of f is a continuous map $g\colon Y \to X$ such that $f \circ g = id_Y$.

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The following facts hold for the topological spaces $\mathbf{B}$ *and* $\mathbf{A}_\infty$*:*

1. *There exists a section* $\sigma\colon \mathbf{A}_\infty \to \mathbf{B}$ *of* π *such that* $\sigma(\mathbf{A}_\infty)$ *is dense in* $\mathbf{B}$*;*
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- $\sigma([a]_\sim) = a$: π admits many sections (depending on the cardinality of the fiber $\pi^{-1}([a])$).

Interior preserving map

Lemma

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Theorem

Let $\mathbf{B} \in \mathcal{IBSL}$ *carrying a faithful state. T. f. a. e.:*

1. $\mathbf{B} = \mathbf{A}_\infty$;
2. $\pi\colon \mathbf{B} \to \mathbf{A}_\infty$ *is interior preserving;*
3. $\sigma(\mathbf{A}_\infty)$ *is open (closed) in* $\mathbf{B}$*, for every section* $\sigma\colon \mathbf{A}_\infty \to \mathbf{B}$*.*

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Theorem (Topological characterisation of states)

Let s be a faithful state over $\mathbf{B}$ and $t\colon \mathbf{B} \to [0,1]$ a continuous map such that $t \circ \sigma = \Phi(s)$, for any section $\sigma\colon \mathbf{A}_\infty \to \mathbf{B}$. Then $t = s$.

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Thank you!

A different definition

Replace the additivity condition by:

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Let $\mathbf{B} \in \mathcal{IBSL}$. Then the following are equivalent:

1. $s\colon \mathbf{B} \to [0,1]$ *satisfies* $s(1) = 1$ *and condition* (1)*;*
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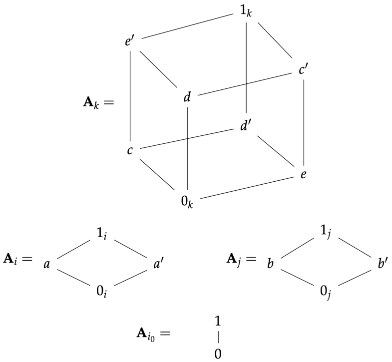
- Only the elements in $\mathbf{A}_{i_0}$ count for probability.



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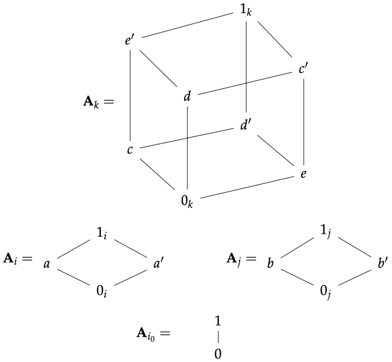
A logical motivation



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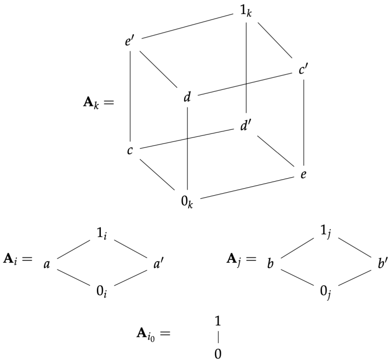

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- a, b not compatible when $a \wedge b \in F'$.