Extending Stone-Priestley duality along full embeddings

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based on joint work with A. L. Suarez

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9 - 13 June

1. The spatial-sober duality

   (and its restriction to duality for bounded distributive lattices)

2. The categories of Pervin spaces and of Frith frames
3. Extending Stone-Priestley duality along full embeddings
4. The bitopological point of view

$$\mathbf{Top} \underset{\Omega}{\overset{\Sigma}{\rightleftarrows}} \mathbf{Frm}^{op} \quad (\Omega \dashv \Sigma)$$

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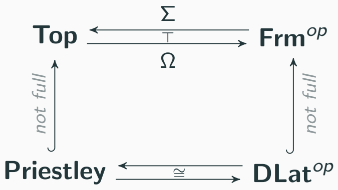

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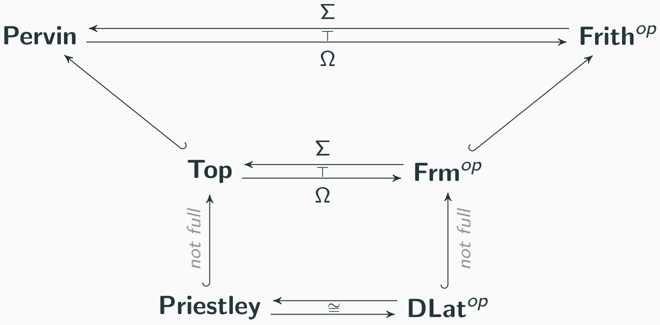

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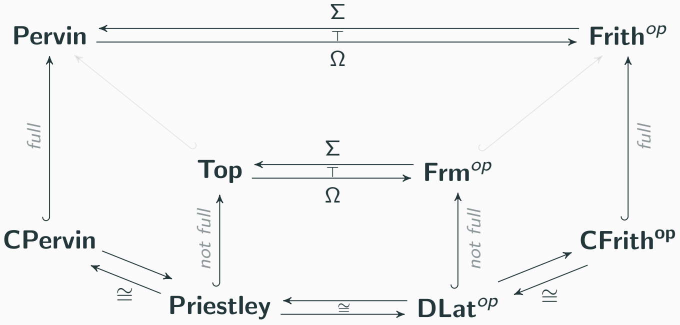

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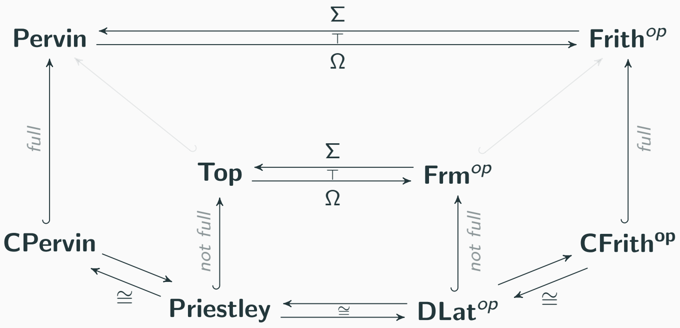

The spatial-sober duality

Frames

A *frame* is a complete lattice L satisfying

$$a \wedge \bigvee_{i \in I} b_i = \bigvee (a \wedge b_i).$$

Frame homomorphisms preserve finite meets and arbitrary joins.

Topological spaces

and continuous functions.

The spatial-sober duality

- $\Omega : \mathbf{Top} \to \mathbf{Frm}^{op}$

$$\Omega(X) := (\{\text{open subsets of } X\}, \subseteq) \qquad \Omega(X \xrightarrow{f} Y) := (\Omega(Y) \xrightarrow{f^{-1}} \Omega(X))$$

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$$\Sigma(L) := \{\text{c. p. filters of } L\} \qquad \Sigma(L \xrightarrow{h} M) := (\Sigma(M) \xrightarrow{h^{-1}} \Sigma(L))$$

$$\widehat{a} := \{F \mid a \in F\}, \quad a \in L$$

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We have an adjunction $\Omega : \mathbf{Top} \leftrightarrows \mathbf{Frm}^{op} : \Sigma$ which restricts and co-restricts to a duality between sober spaces and spatial frames.

– Spatial frame: a frame of the form $\Omega(X)$ for some topological space X.

– Sober space: a space that is completely determined by its set of open subsets.

Bounded distributive lattices seen as coherent frames

A coherent frame is a frame L whose set of compact elements $K(L)$ is closed under finite meets (thus, a sublattice) and join-dense in L.

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  If $h : C \to D$ is a lattice homomorphism, $\mathrm{Idl}(h) : (J \in \mathrm{Idl}(C)) \mapsto \langle h[J] \rangle_{\mathrm{Idl}}$ is a coherent homomorphism.

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- If L is a coherent frame, $K(L)$ is a bounded distributive lattice.
  If $h : L \to M$ is a coherent homomorphism, the restriction and co-restriction $K(h) : K(L) \to K(M)$ of h is a lattice homomorphism.

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These assignments define an equivalence of categories **DLat** $\cong$ **CohFrm**

Spectral spaces

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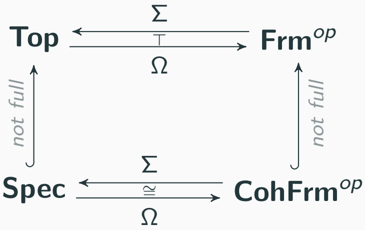
The adjunction $\Omega : \mathbf{Top} \leftrightarrows \mathbf{Frm}^{op} : \Sigma$ restricts and co-restricts to an equivalence

$$\mathbf{Spec} \cong \mathbf{CohFrm}^{op}$$

Stone-Priestley duality for bounded distributive lattices

$$\mathbf{Top} \underset{\Omega}{\overset{\Sigma}{\rightleftarrows}} \mathbf{Frm}^{op} \quad (\Sigma \dashv \Omega)$$

Stone-Priestley duality for bounded distributive lattices



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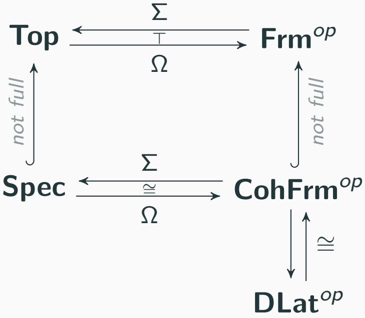

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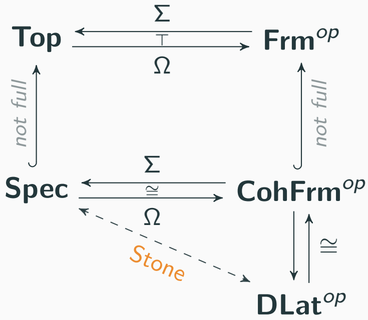

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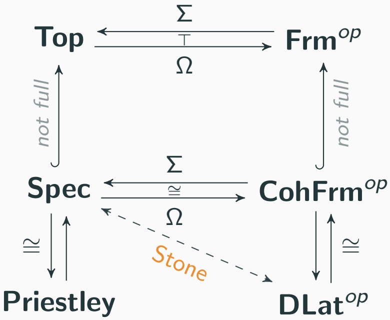

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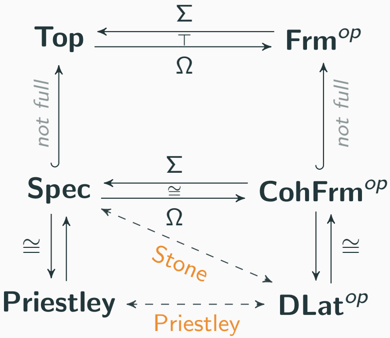

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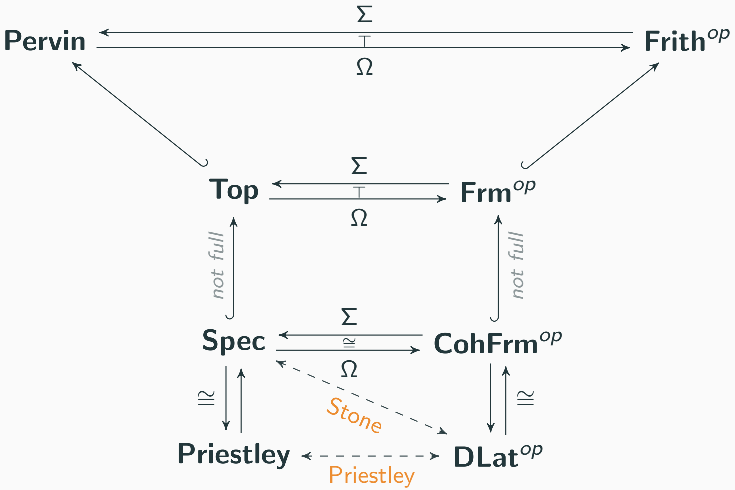

Pervin spaces

What are the quasi-uniformizable topological spaces?

Every topology comes from a quasi-uniformity!

Every topology comes from a transitive and totally bounded quasi-uniformity.

Pervin spaces

A Pervin space is a pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S} \subseteq \mathcal{P}(X)$ is a bounded sublattice.

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A morphism of Pervin spaces $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ is a map $f : X \to Y$ such that for every $T \in \mathcal{T}$, we have $f^{-1}(T) \in \mathcal{S}$.

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There is a full embedding **Top** $\hookrightarrow$ **Pervin**, $\quad (X, \tau) \mapsto (X, \tau)$.

Theorem (Pin, 2017)

The category of transitive and totally bounded quasi-uniform spaces is equivalent to the category **Pervin** *of Pervin spaces.*

Frith frames

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There is a full embedding $\mathbf{Frm} \hookrightarrow \mathbf{Frith}$, $\quad L \mapsto (L, L)$.

Theorem (B., Suarez)

The category **Frith** *of Frith frames is a full subcategory of the category of transitive and totally bounded quasi-uniform frames.*

The Frith-Pervin adjunction

- $\Omega : \textbf{Pervin} \to \textbf{Frith}^{op}$

  $\Omega(X, \mathcal{S}) := (\langle \mathcal{S} \rangle_{\textbf{Frm}},\ \mathcal{S})$

  $\Omega((X, \mathcal{S}) \xrightarrow{f} (Y, \mathcal{T})) := (\Omega(Y, \mathcal{T}) \xrightarrow{f^{-1}} \Omega(X, \mathcal{S}))$

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- $\Sigma : \textbf{Frith}^{op} \to \textbf{Pervin}$

$$\Sigma(L, S) := (\Sigma(L),\ \{\widehat{s} \mid s \in S\}) \qquad (\widehat{s} := \{F \mid s \in F\})$$

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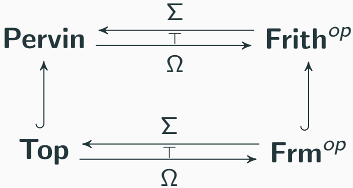
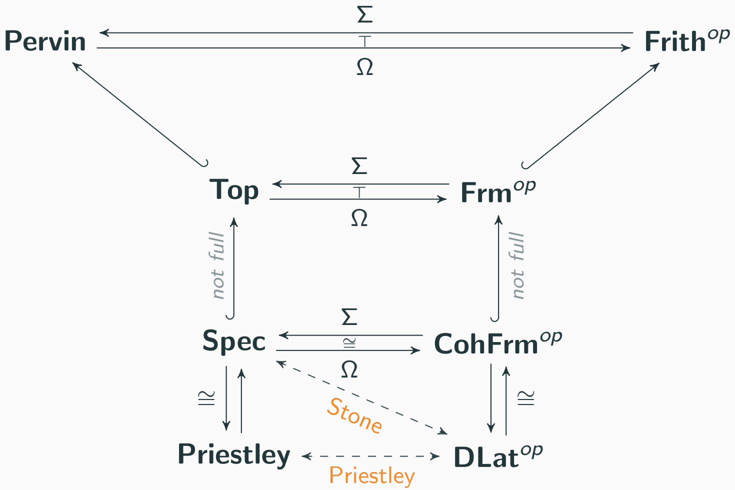


Stone-Priestley duality for bounded distributive lattices



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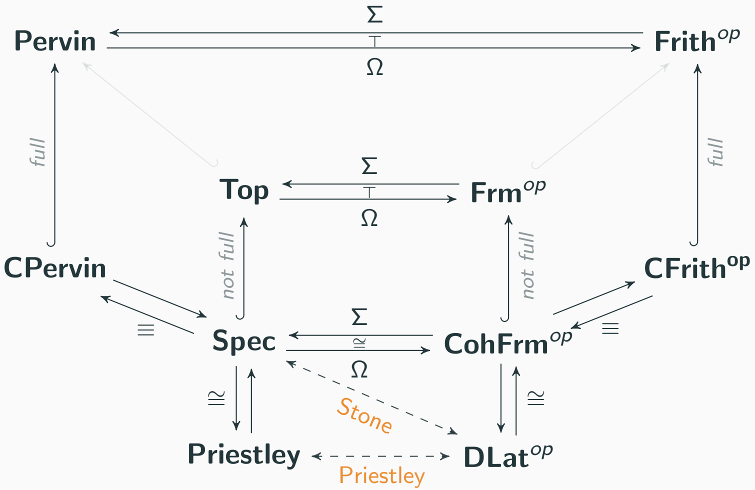

Symmetric Frith frames

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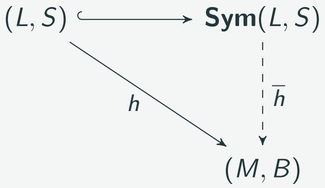
Proposition (B., Suarez)

Symmetric Frith frames form a full reflective subcategory of **Frith**.

That is, for every Frith frame (L, S), there exists a symmetric Frith frame

$$\mathbf{Sym}(L, S) = (\mathcal{C}(L, S), \langle S \rangle_{\mathbf{BA}}),$$

called the symmetrization of (L, S), such that for every $h : (L, S) \to (M, B)$ with (M, B) symmetric there is a unique $\overline{h}$ making the following diagram commute:



Completion of Frith frames

- A symmetric Frith frame (L, B) is complete if every dense surjection[1] $(M, C) \twoheadrightarrow (L, B)$ with (M, C) symmetric is an isomorphism.

[1] $h : (M, C) \twoheadrightarrow (L, B)$ is a dense surjection if $(h(a) = 0 \implies a = 0)$ and $h[C] = B$.

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Corollary

The categories of coherent frames and of complete Frith frames are isomorphic.

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Completion of Frith frames

Proof's idea:

- A completion of (L, S) is a dense surjection $(M, T) \twoheadrightarrow (L, S)$ with (M, T) complete.

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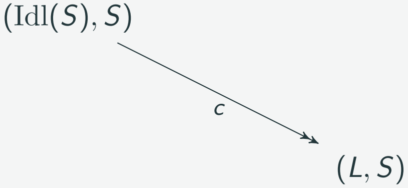

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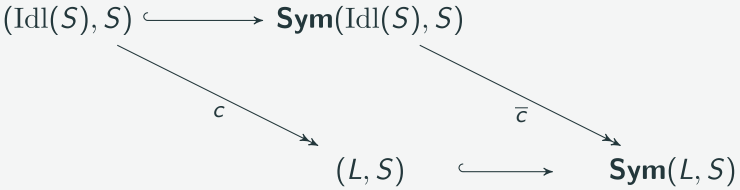

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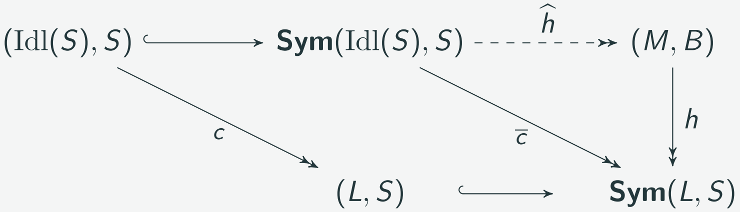

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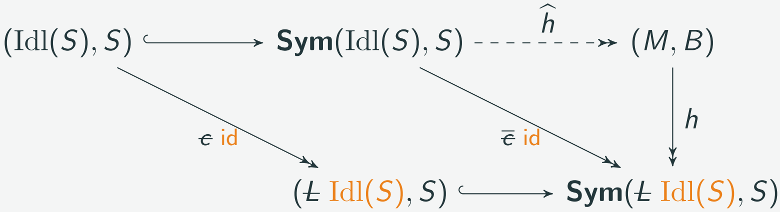

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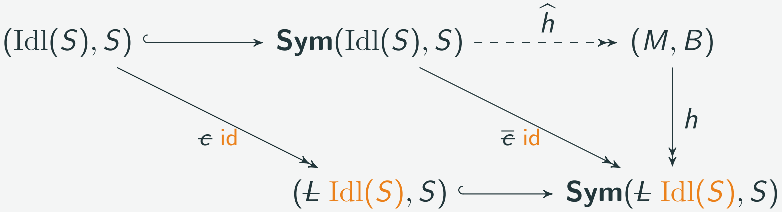


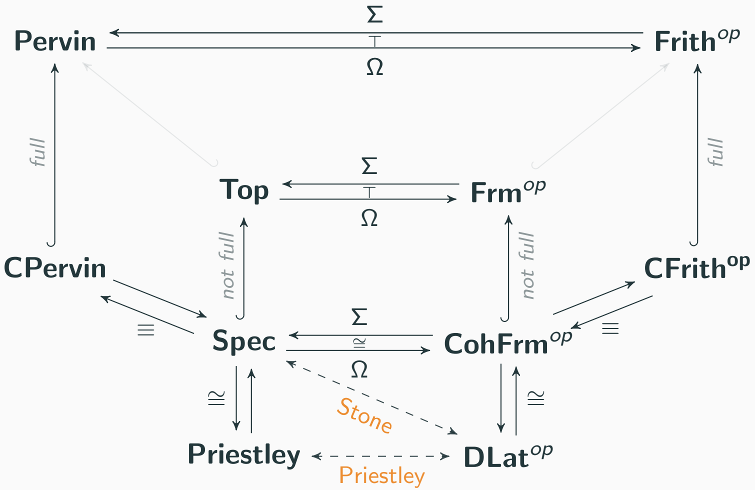
Therefore, h is one-one, thus an isomorphism. □

Completion of Pervin spaces

Theorem (Gehrke, Grigorieff, Pin, 2010; Pin, 2017)

The categories of spectral spaces and of complete T_0 Pervin spaces are isomorphic.

Stone-Priestley duality for bounded distributive lattices



The bitopological point of view

Bitopological spaces and biframes

A bitopological space is a triple (X, τ_1, τ_2), where τ_i is a topology on X.

A biframe is a triple (L, L_1, L_2) of frames st $L_i \leq L$ and $L = \langle L_1 \cup L_2 \rangle_{\mathbf{Frm}}$.

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- $\Omega : \mathbf{biTop} \to \mathbf{biFrm}^{op}$

  $\Omega(X, \tau_1, \tau_2) := (\tau_1 \vee \tau_2,\ \tau_1,\ \tau_2)$

- $\Sigma : \mathbf{biFrm}^{op} \to \mathbf{biTop}$

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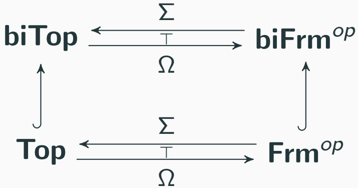
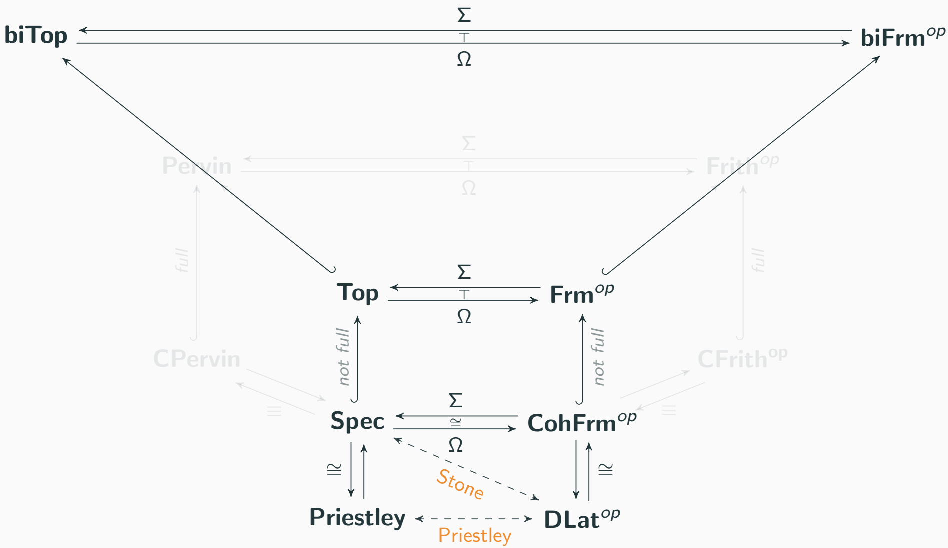


Pairwise Stone spaces are dual to bounded distributive lattices



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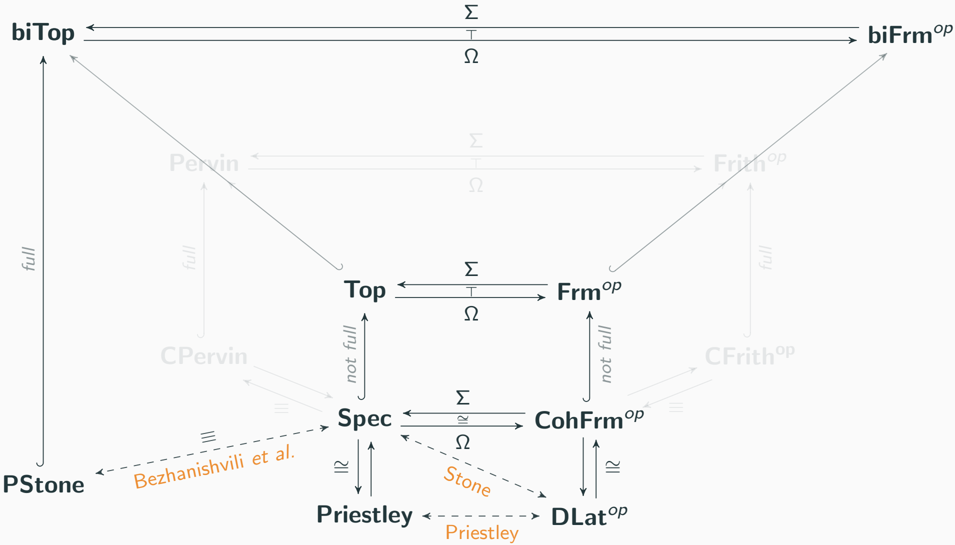

The Skula functor

If $(X, \mathcal{S})$ is a Pervin space, $(X, \langle \mathcal{S} \rangle_{\mathbf{Top}}, \langle \{U^c \mid U \in \mathcal{S}\} \rangle_{\mathbf{Top}})$ is a bispace.

This defines a full embedding

$$\mathbf{Sk}_{\text{Pervin}} : \mathbf{Pervin} \hookrightarrow \mathbf{biTop}$$

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Proposition (Bezhanishvili et al., 2010)

A bitopological space is a pairwise Stone space (i.e., *pairwise compact, pairwise Hausdorff, and pairwise* 0*-dimensional*) if and only if it is of the form $\textbf{Sk}_{\text{Pervin}}(X, \mathcal{S})$ for some complete T_0 Pervin space $(X, \mathcal{S})$.

The bitopological space (X, τ_1, τ_2) is pairwise...

- compact if every cover of $\tau_1 \cup \tau_2$ has a finite refinement;
- Hausdorff if for every $x \neq y$, $\exists U_i \in \tau_i$ disjoint : $x \in U_k$, $y \in U_\ell$, $\{k, \ell\} = \{1, 2\}$;
- 0-dimensional if $\{U \in \tau_k \mid U^c \in \tau_\ell\}$ is a basis for τ_k, where $\{k, \ell\} = \{1, 2\}$.

The Skula functor

If (L, S) is a Frith frame, $(\mathcal{C}(L, S),\ L,\ \langle\{s^c \mid s \in S\}\rangle_{\mathbf{Frm}})$ is a biframe.

This defines a full embedding

$$\mathbf{Sk}_{\mathrm{Frith}} : \mathbf{Frith} \hookrightarrow \mathbf{biFrm}$$

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Proposition (B., Suarez)

A biframe is compact and 0-dimensional if and only if it is of the form $\mathbf{Sk}_{\mathrm{Frith}}(L, S)$ for some complete Frith frame (L, S).

The biframe (L, L_1, L_2) is:

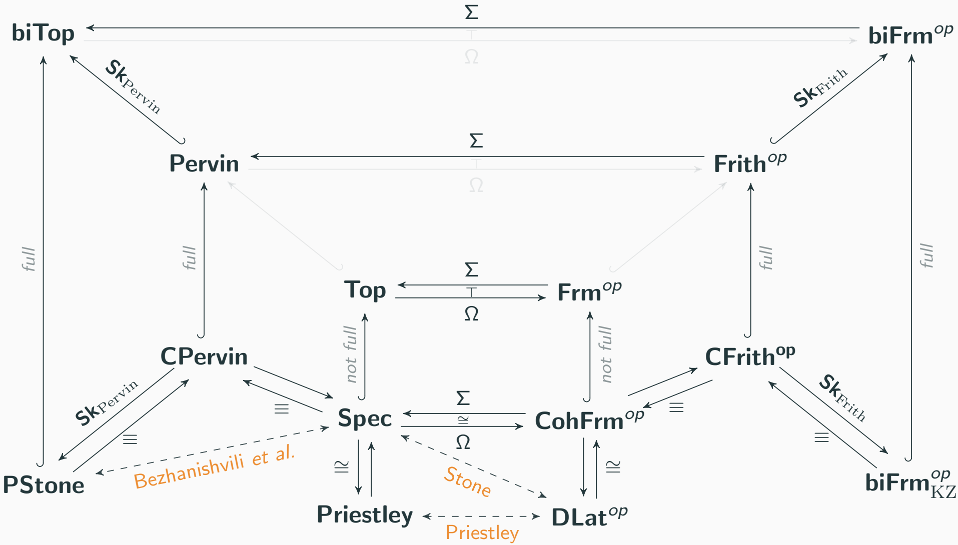
- compact if L is compact;
- 0-dimensional if $L_i = \langle\{a \in L_i \text{ complemented} \mid \neg a \in L_j\}\rangle_{\mathbf{Frm}}$ with $\{i, j\} = \{1, 2\}$.

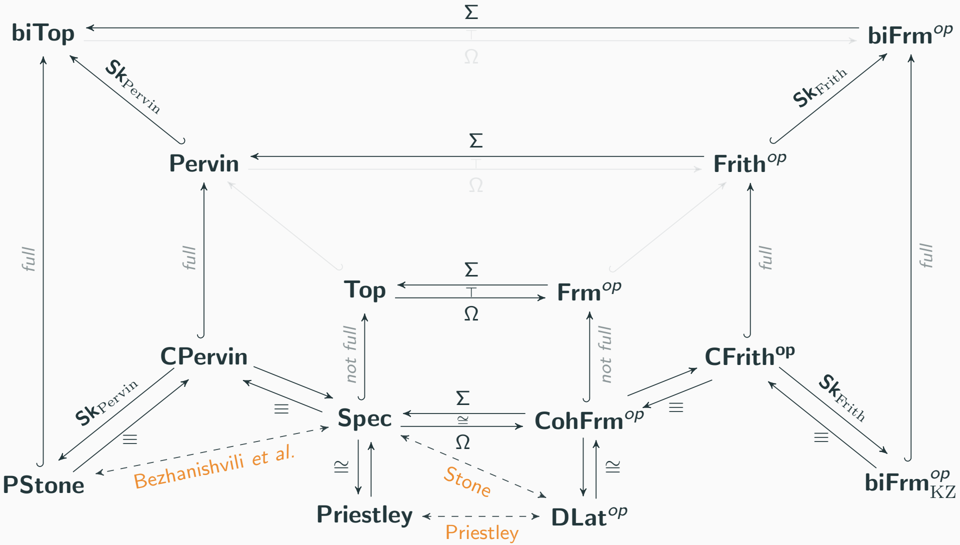
The Skula functor

Proposition (B., Suarez)

The following square commutes up to natural isomorphism.

$$
\begin{array}{ccc}
\mathbf{Frith}^{op} & \xhookrightarrow{\mathsf{Sk}_{\mathrm{Frith}}} & \mathbf{biFrm}^{op} \\
\Sigma \downarrow & & \downarrow \Sigma \\
\mathbf{Pervin} & \xhookrightarrow[\mathsf{Sk}_{\mathrm{Pervin}}]{} & \mathbf{biTop}
\end{array}
$$


biTop
biFrm^{op}
Σ
⊤
Ω
$\mathsf{Sk}_{\mathrm{Pervin}}$
$\mathsf{Sk}_{\mathrm{Frith}}$
Pervin
Frith^{op}
full
full
full
full
Top
Frm^{op}
not full
not full
CPervin
CFrith^{op}
Spec
CohFrm^{op}
≅
≡
PStone
biFrm$^{op}_{\mathrm{KZ}}$
Bezhanishvili et al.
Stone
Priestley
DLat^{op}
Priestley



Thank you for your attention!