Covering versus partitioning with Polish spaces

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BLAST
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Covering and partitioning numbers

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\text{cov}(X) = \min\{|C| : C \text{ is a covering of } X \text{ with Polish spaces}\},
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\text{par}(X) = \min\{|\mathcal{P}| : \mathcal{P} \text{ is a partition of } X \text{ into Polish spaces}\}.
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Motivating question:

Is it possible to have $\text{cov}(X) < \text{par}(X)$?
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**Theorem**

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Furthermore, large cardinal hypotheses are required for obtaining this inequality:

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*If* $\text{cov}(X) < \text{par}(X)$ *for any completely metrizable space* $X$, *then* $0^+$ *exists.*

We aim to sketch some of the main ideas involved in proving these two theorems, beginning with the second. The first step is a ZFC theorem concerning spaces of weight $< \aleph_\omega$. 
Partitioning spaces of uncountable weight

Lemma

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Proof.

Let $\mathcal{B}$ be a basis for $X$ such that $|\mathcal{B}| = w(X) = \kappa$, and every point of $X$ is contained in only countably many members of $\mathcal{B}$. 
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Let $B$ be a basis for $X$ such that $|B| = w(X) = \kappa$, and every point of $X$ is contained in only countably many members of $B$. Write $B = \{ U_\alpha : \alpha < \kappa \}$. For each $\alpha < \kappa$, let

$X_\alpha = \{ x \in X : \text{if } x \in U_\beta \text{ then } \beta < \alpha \}$

and

$Y_\alpha = X_\alpha \setminus \bigcup_{\beta < \alpha} X_\beta.$
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Proof.

Let \( \mathcal{B} \) be a basis for \( X \) such that \( |\mathcal{B}| = w(X) = \kappa \), and every point of \( X \) is contained in only countably many members of \( \mathcal{B} \). Write \( \mathcal{B} = \{ U_\alpha : \alpha < \kappa \} \). For each \( \alpha < \kappa \), let

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X_\alpha = \{ x \in X : \text{if } x \in U_\beta \text{ then } \beta < \alpha \} \quad \text{and} \quad Y_\alpha = X_\alpha \setminus \bigcup_{\beta < \alpha} X_\beta.
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Because \( \text{cf}(\kappa) > \omega \), and because of our choice of \( \mathcal{B} \), every \( x \in X \) is in some \( X_\alpha \), and therefore the \( Y_\alpha \) partition \( X \).
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Because $\text{cf}(\kappa) > \omega$, and because of our choice of $\mathcal{B}$, every $x \in X$ is in some $X_\alpha$, and therefore the $Y_\alpha$ partition $X$. Clearly each $X_\alpha$, and therefore each $Y_\alpha$, has weight $< \kappa$. 

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It is now straightforward to show the $Y_\alpha$ are completely metrizable:
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It is now straightforward to show the \( Y_\alpha \) are completely metrizable:

If \( \alpha = \beta + 1 \), then \( Y_\alpha = X_\alpha \setminus X_\beta \) is the difference of two closed sets, hence \( G_\delta \).
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If $\alpha$ has countable cofinality, write $\alpha = \sup \langle \beta_n : n \in \omega \rangle$, and then $Y_\alpha = X_\alpha \setminus \bigcup_{n \in \omega} X_\beta_n$ is again $G_\delta$. 
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If $\alpha$ has uncountable cofinality, then $Y_\alpha = \emptyset$, because $X_\alpha = \bigcup_{\beta < \alpha} X_\beta$ by our choice of $\mathcal{B}$. 

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The previous theorem, plus a simple induction argument, shows $\text{par}(X) \leq w(X)$ whenever $w(X) < \aleph_\omega$. It is not difficult to prove that $w(X) \leq \text{cov}(X) \leq \text{par}(X)$ for any $X$. 
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**Corollary**

Suppose $\text{cov}(X) < \text{par}(X)$ for some completely metrizable space $X$. If $\kappa$ is the minimum possible weight of such a space $X$, then $\kappa$ is a singular cardinal with $\text{cf}(\kappa) = \omega$. Furthermore, $\aleph_\omega \leq \kappa < c$. 
Using "L-like" principles, it is possible to push this inductive argument past $\aleph_\omega$ and get similar results at higher cardinals.
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**Theorem**

Assume $\square_\kappa$ holds for all singular cardinals $\kappa < c$, then $\text{cov}(X) = \text{par}(X)$ for all completely metrizable spaces $X$. 

**Corollary**

If $\text{cov}(X) < \text{par}(X)$ for some completely metrizable space $X$, then $0^+$ exists.
SSH and $\square$

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**Theorem**

Assume $\square_\kappa$ holds for all singular cardinals $\kappa < \mathfrak{c}$, and $(SSH)$: if $\kappa$ is a singular cardinal with cofinality $\omega$, then the poset $([\kappa]^\omega, \subseteq)$ has cofinality $\kappa^+$. 

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If $\text{cov}(X) < \text{par}(X)$ for some completely metrizable space $X$, then $0^\dagger$ exists.
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Chang’s Conjecture for $\aleph_\omega$, which is abbreviated by writing $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$, states:
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unary predicate A, if \mid \mathcal{M} \mid = \aleph_{\omega+1} and \mid A \mid = \aleph_\omega then there is an
elementary submodel \mathcal{M}' \subset \mathcal{M} such that \mid \mathcal{M}' \mid = \aleph_1 and
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Levinsky, Magidor, and Shelah proved $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$ is consistent relative to a large cardinal hypothesis a little stronger than a huge cardinal.
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This was improved recently by Eskew and Hayut, who showed $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$ is consistent relative to a huge cardinal.
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Suppose \(M\) is a collection of \(\geq \aleph_{\omega+1}\) structures (molecules), each built from countably many members of an \(\aleph_\omega\)-sized set \(A\) (atoms).
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how to get $\text{cov}(X) < \text{par}(X)$

**Theorem**

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Proof sketch.

Begin with a model of $\text{GCH} + (\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$. (This is the part that requires a huge cardinal.)
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In the extension, we have $\text{cov}(D^\omega) = \aleph_{\omega+1}$. (This is where the GCH of the ground model comes in.)
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Forcing with a ccc poset preserves $(\aleph_{\omega+1}, \aleph_\omega) \rightarrowcreg (\aleph_1, \aleph_0)$, so $(\aleph_{\omega+1}, \aleph_\omega) \rightarrowcreg (\aleph_1, \aleph_0)$ holds in the extension.
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Working in the extension, suppose \( \mathcal{P} \) is a partition of \( D^\omega \).
how to get $\text{cov}(X) < \text{par}(X)$

**Theorem**

Let $D$ be the discrete space of cardinality $\aleph_\omega$. It is consistent, relative to a huge cardinal, that $\text{cov}(D^\omega) < \text{par}(D^\omega)$.

**Proof sketch.**

As $\text{par}(D^\omega) \geq \text{cov}(D^\omega) = \aleph_{\omega+1}$, $P$ is a collection of $\geq \aleph_{\omega+1}$ Polish spaces.
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Covering versus partitioning with Polish spaces
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Will Brian

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Open questions

**Question**

Can one find more precise bounds on the consistency strength of the statement “There is a completely metrizable space $X$ with $\text{cov}(X) < \text{par}(X)$”? Does a supercompact suffice?
Open questions

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The end

Thank you for listening