Covering versus partitioning with Polish spaces

Will Brian

University of North Carolina at Charlotte

BLAST

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Covering and partitioning numbers

Given a completely metrizable space X, define

$$\mathfrak{cov}(X) = \min\{|\mathcal{C}| : \mathcal{C} \text{ is a covering of } X \text{ with Polish spaces}\},$$

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Motivating question:

Is it possible to have $\mathfrak{cov}(X) < \mathfrak{par}(X)$?

The main results

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We aim to sketch some of the main ideas involved in proving these two theorems, beginning with the second. The first step is a ZFC theorem concerning spaces of weight $< \aleph_\omega$.

Partitioning spaces of uncountable weight

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If X is completely metrizable and $w(X) = \kappa$ has uncountable cofinality, then X can be partitioned into $\leq \kappa$ completely metrizable spaces of strictly smaller weight.

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$$X_\alpha = \{x \in X : \text{if } x \in U_\beta \text{ then } \beta < \alpha\} \quad \text{and} \quad Y_\alpha = X_\alpha \setminus \bigcup_{\beta<\alpha} X_\beta.$$

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Because $\mathrm{cf}(\kappa) > \omega$, and because of our choice of $\mathcal{B}$, every $x \in X$ is in some X_α, and therefore the Y_α partition X.

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Because $\mathrm{cf}(\kappa) > \omega$, and because of our choice of $\mathcal{B}$, every $x \in X$ is in some X_α, and therefore the Y_α partition X. Clearly each X_α, and therefore each Y_α, has weight $< \kappa$.

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If $\langle x_n : n \in \omega \rangle$ is a sequence in X_α with limit p, then every neighborhood U_β of p is also a neighborhood of some x_n, and therefore $\beta < \alpha$ (as $x_n \in X_\alpha$).

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If $\langle x_n : n \in \omega \rangle$ is a sequence in X_α with limit p, then every neighborhood U_β of p is also a neighborhood of some x_n, and therefore $\beta < \alpha$ (as $x_n \in X_\alpha$). So $\beta < \alpha$ whenever $U_\beta \ni p$, which means that $p \in X_\alpha$.

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If α has countable cofinality, write $\alpha = \sup\langle \beta_n : n \in \omega \rangle$, and then $Y_\alpha = X_\alpha \setminus \bigcup_{n\in\omega} X_{\beta_n}$ is again G_δ.

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If α has uncountable cofinality, then $Y_\alpha = \emptyset$, because $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ by our choice of $\mathcal{B}$. $\square$

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The previous theorem, plus a simple induction argument, shows $\mathfrak{par}(X) \leq w(X)$ whenever $w(X) < \aleph_\omega$. It is not difficult to prove that $w(X) \leq \mathfrak{cov}(X) \leq \mathfrak{par}(X)$ for any X. $\square$

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Corollary

Suppose $\mathfrak{cov}(X) < \mathfrak{par}(X)$ for some completely metrizable space X. If κ is the minimum possible weight of such a space X, then κ is a singular cardinal with $\mathrm{cf}(\kappa) = \omega$. Furthermore, $\aleph_\omega \leq \kappa < \mathfrak{c}$.

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Then $\mathfrak{cov}(X) = \mathfrak{par}(X)$ for all completely metrizable spaces X.

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Then $\mathfrak{cov}(X) = \mathfrak{par}(X)$ for all completely metrizable spaces X.

Corollary

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$(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$

Chang's Conjecture for $\aleph_\omega$, which is abbreviated by writing $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$, states:

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For every model M for a countable language $\mathcal{L}$ that contains a unary predicate A, if $|M| = \aleph_{\omega+1}$ and $|A| = \aleph_\omega$ then there is an elementary submodel $M' \prec M$ such that $|M'| = \aleph_1$ and $|M' \cap A| = \aleph_0$.

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Levinsky, Magidor, and Shelah proved $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ is consistent relative to a large cardinal hypothesis a little stronger than a huge cardinal.

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This was improved recently by Eskew and Hayut, who showed $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ is consistent relative to a huge cardinal.

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For example, suppose X is a completely metrizable space of weight $\aleph_\omega$. We could take A to be an $\aleph_\omega$-sized basis for X, and M to be some $\geq \aleph_{\omega+1}$-sized collection of Polish subspaces of X.

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Each member of M is "built" from a countable subset of A (because each member of M is a second countable G_δ).
So, $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ implies that uncountably many members of M are defined from some countable $A_0 \subseteq A$.

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Begin with a model of GCH + $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$. (This is the part that requires a huge cardinal.)

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In the extension, we have $\mathfrak{cov}(D^\omega) = \aleph_{\omega+1}$. (This is where the GCH of the ground model comes in.)
Forcing with a ccc poset preserves $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$, so $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ holds in the extension.

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Forcing with a ccc poset preserves $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$, so $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ holds in the extension.

Working in the extension, suppose $\mathcal{P}$ is a partition of D^ω.

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As $\mathfrak{par}(D^\omega) \geq \mathfrak{cov}(D^\omega) = \aleph_{\omega+1}$, $\mathcal{P}$ is a collection of $\geq \aleph_{\omega+1}$ Polish spaces.

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As $\mathfrak{par}(D^\omega) \geq \mathfrak{cov}(D^\omega) = \aleph_{\omega+1}$, $\mathcal{P}$ is a collection of $\geq \aleph_{\omega+1}$ Polish spaces. Applying $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$, there are uncountably many members of $\mathcal{P}$ that are all "built from" the same countable collection of basic open sets.

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What is the consistency strength of the failure of SSH*?*

The end

Thank you for listening