Esakia's theorem in the monadic setting

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Important translations:

- Double negation translation of classical logic into intuitionistic logic (Glivenko)
- Translation of intuitionistic logic into modal logic (Gödel)

Gödel translation

In 1933 Gödel proposed a translation of the intuitionistic propositional calculus IPC into the modal logic S4.

The Gödel translation

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\begin{aligned}
\mathsf{T}(\bot) &= \bot \\
\mathsf{T}(p) &= \Box p \\
\mathsf{T}(\varphi \wedge \psi) &= \mathsf{T}(\varphi) \wedge \mathsf{T}(\psi) \\
\mathsf{T}(\varphi \vee \psi) &= \mathsf{T}(\varphi) \vee \mathsf{T}(\psi) \\
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Theorem (McKinsey-Tarski 1948)

T *translates* IPC *into* S4 *fully and faithfully, i.e.*

$$\mathsf{IPC} \vdash \varphi \qquad \textit{iff} \qquad \mathsf{S4} \vdash T(\varphi)$$

for any intuitionistic formula φ*.*

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- Dutch (Troelstra, de Jongh, etc.)
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Dummett and Lemmon started to investigate the Gödel translation of superintuitionistic logics into normal extensions of S4.

Definition

Let L be a superintuitionistic logic and M a normal extension of S4.

We call L the intuitionistic fragment of M and M a modal companion of L if

$$\mathsf{L} \vdash \varphi \qquad \text{iff} \qquad \mathsf{M} \vdash \mathsf{T}(\varphi)$$

for any intuitionistic formula φ.

Each consistent superintuitionistic logic has many modal companions.

The least modal companion of IPC is S4. The greatest one is Grz.

Definition

Let $\mathsf{Grz} := \mathsf{S4} + \mathsf{grz}$ where

$$\mathsf{grz} := \Box(\Box(p \to \Box p) \to p) \to p$$

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Theorem (Esakia's theorem 1976)

Grz *is the greatest modal companion of* IPC.

Maksimova and Rybakov introduced the mappings ρ, τ, and σ.

Theorem

Let L *be a superintuitionistic logic and* M *a normal extension of* S4*.*

- $\rho\mathsf{M} := \{\varphi \mid \mathsf{T}(\varphi) \in \mathsf{M}\}$ *is the intuitionistic fragment of* M*.*
- $\tau\mathsf{L} := \mathsf{S4} + \{\mathsf{T}(\varphi) \mid \varphi \in \mathsf{L}\}$ *is the least modal companion of* L*.*
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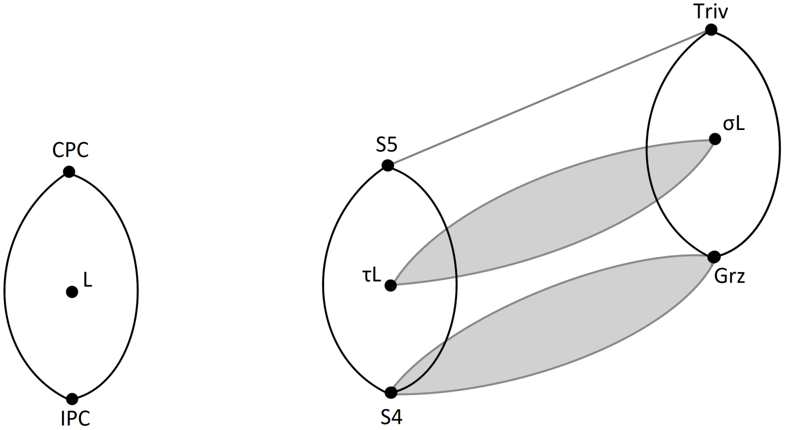
Theorem (Blok-Esakia 1976)

σ is a lattice isomorphism between the lattice of superintuitionistic logics and the lattice of normal extensions of Grz*.*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L
IPC
Triv
σL
S5
τL
Grz
S4

Gödel translation in the predicate setting

Rasiowa and Sikorski extended the Gödel translation to the predicate setting as follows:

$$\begin{aligned} \mathsf{T}(\forall x \varphi) &= \Box \forall x \mathsf{T}(\varphi) \\ \mathsf{T}(\exists x \varphi) &= \exists x \mathsf{T}(\varphi) \end{aligned}$$

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Theorem (Rasiowa-Sikorski 1953)

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$$\forall x(P(x) \to \exists x Q(x))$$

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$$\forall(p \to \exists q)$$

Therefore, monadic fragments can be treated like propositional bimodal logics with modalities $\forall$, $\exists$.

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The predicate Gödel translation restricts to a full and faithful translation of MIPC into MS4.

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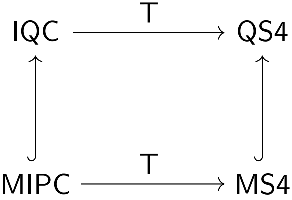

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$$\begin{aligned} \mathsf{T}(\forall\varphi) &= \Box\forall\mathsf{T}(\varphi) \\ \mathsf{T}(\exists\varphi) &= \exists\mathsf{T}(\varphi) \end{aligned}$$

ρM, τL, and σL can be defined similarly in the monadic setting.

Semantics

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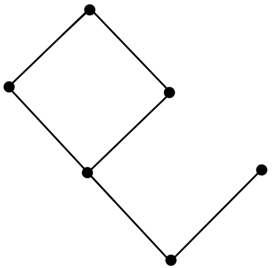
For convenience, we will mainly concentrate on relational semantics.

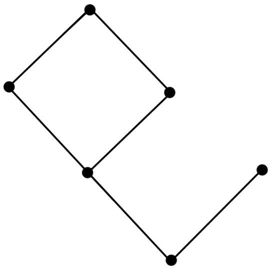
Definition

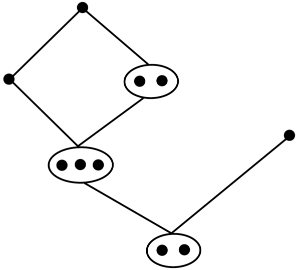
- An IPC-frame is a pair $\mathfrak{F} = (X, R)$ where X is a set and R is a partial order.



Definition

- An IPC-frame is a pair $\mathfrak{F} = (X, R)$ where X is a set and R is a partial order.
- An S4-frame is a pair $\mathfrak{G} = (X, R)$ where X is a set and R is a reflexive and transitive relation, i.e. a quasi-order.





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Given an S4-frame $\mathfrak{G} = (X, R)$ we can define an equivalence relation $\sim$ on X by setting

$$x \sim y \quad \text{iff} \quad xRy \text{ and } yRx.$$

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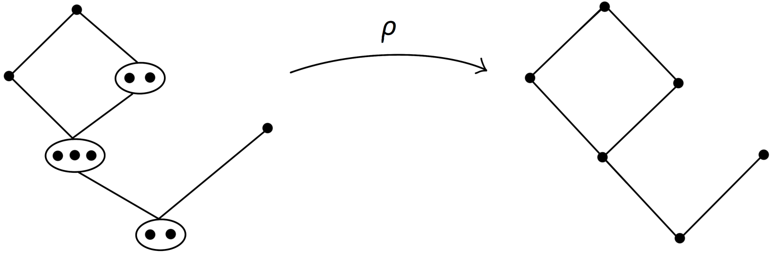
Let $\rho\mathfrak{G}$ be the IPC-frame obtained by identifying each R-cluster of $\mathfrak{G}$ to a point. We call $\rho\mathfrak{G}$ the skeleton of $\mathfrak{G}$.



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$$\rho\mathfrak{G} \models \varphi \quad \textit{iff} \quad \mathfrak{G} \models T(\varphi)$$
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$$\mathfrak{F} \vDash \varphi \quad \textit{iff} \quad \sigma\mathfrak{F} \vDash T(\varphi)$$

Moreover, if $\mathfrak{G} \vDash \varphi$, then $\sigma\rho\mathfrak{G} \vDash \varphi$.

Theorem

A finite S4*-frame validates* grz *iff it is a poset.*

Thus, the finite Grz-frames are exactly the ones of the form $\sigma\mathfrak{F}$ for some finite IPC-frame $\mathfrak{F}$.

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- $\sigma\mathfrak{F} = \sigma\rho\mathfrak{G}$ is an M-frame that validates grz.
- Thus, each finite Grz-frame is an M-frame.
- Since Grz has the finite model property, M $\subseteq$ Grz.

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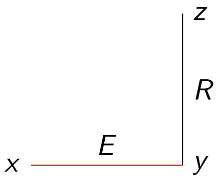

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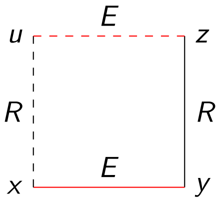

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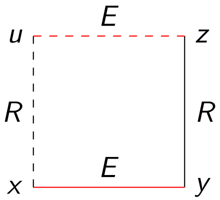
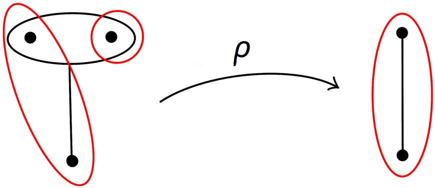


Definition

MS4-frames are defined analogously but R is only required to be a quasi-order.

Definition

Let $\mathfrak{G} = (X, R, E)$ be an MS4-frame. The skeleton $\rho\mathfrak{G}$ of $\mathfrak{G}$ is the skeleton of (X, R) equipped with the equivalence relation induced by the join of $\sim$ and E.



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- An S4-algebra is a boolean algebra equipped with a unary operator $\Box$ satisfying Kuratowski's axioms for interior

$$\Box 1 = 1 \qquad \Box(a \wedge b) = \Box a \wedge \Box b \qquad \Box a \leq a \qquad \Box a \leq \Box\Box a$$

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Theorem

We have $\rho\sigma H \cong H$ and that $\sigma\rho B$ embeds into B.
Moreover,

$$H \vDash \varphi \text{ iff } \sigma H \vDash T(\varphi)$$
$$\rho B \vDash \varphi \text{ iff } B \vDash T(\varphi)$$

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Dually: a descriptive MIPC-frame is not a descriptive MS4-frame.

Indeed, in a descriptive MS4-frame,

- if A is clopen, then $E[A]$ is clopen.

However, in a descriptive MIPC-frame,

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Open problem

Does every extension of MIPC have at least one modal companion?

Main results

Negative results

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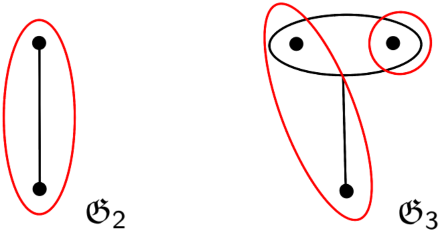

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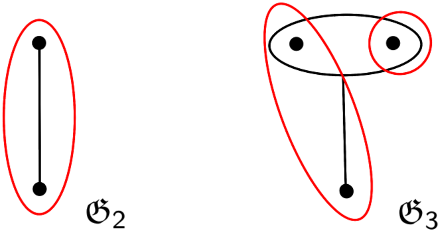

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The logics of $\mathfrak{G}_2$ and $\mathfrak{G}_3$ are incomparable and have the same intuitionistic fragment.

That the logics of $\mathfrak{G}_2$ and $\mathfrak{G}_3$ are incomparable is a consequence of Jónsson's lemma.

Theorem

MGrz *is not the greatest modal companion of* MIPC.

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Sketch:

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- $\mathfrak{G}_2$ validates MGrz and not M, whereas $\mathfrak{G}_3$ validates M and not MGrz. So MGrz and M are incomparable.
- Each finite MIPC-frame is the skeleton of an MGrz-frame and of an M-frame. Thus, both logics are modal companions of MIPC by the finite model property of MIPC.

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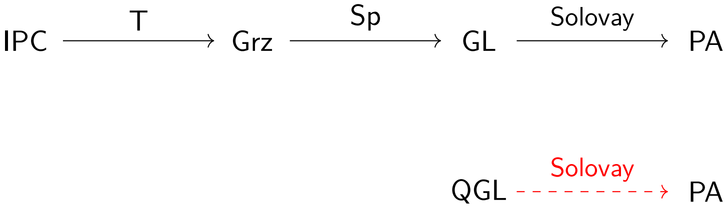
But the Kuroda formula is not a theorem of MIPC.

This is related to a result of Naumov regarding the full predicate case.

Positive results

$$\mathsf{GL} \xrightarrow{\text{Solovay}} \mathsf{PA}$$

$$\mathsf{IPC} \xrightarrow{\quad \mathsf{T} \quad} \mathsf{Grz} \xrightarrow{\quad \mathsf{Sp} \quad} \mathsf{GL} \xrightarrow{\quad \mathsf{Solovay} \quad} \mathsf{PA}$$


IPC
T
Grz
Sp
GL
Solovay
PA
QGL
Solovay
PA

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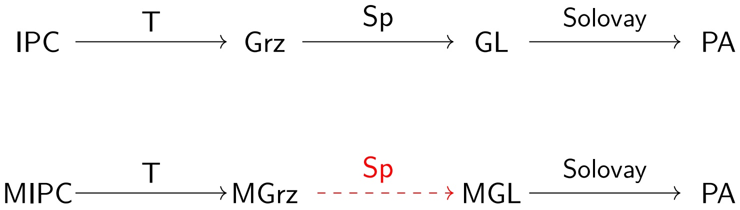
IPC
T
Grz
Sp
GL
Solovay
PA
MIPC
T
MGrz
Sp
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$$\mathsf{M^{+}IPC} \xrightarrow{\quad \mathsf{T} \quad} \mathsf{M^{+}Grz} \xrightarrow{\quad \mathsf{Sp} \quad} \mathsf{MGL} \xrightarrow{\quad \mathsf{Solovay} \quad} \mathsf{PA}$$

Definition

Let

$$\mathsf{MCas} = \forall((p \to \forall p) \to \forall p) \to \forall p.$$

We write

$$\mathsf{M^{+}IPC} := \mathsf{MIPC} + \mathsf{MCas} \quad \text{and} \quad \mathsf{M^{+}Grz} := \mathsf{MGrz} + \mathsf{T}(\mathsf{MCas}).$$

Theorem

M^+Grz *is the greatest modal companion of* M^+IPC.

Theorem

$\mathsf{M}^+\mathsf{Grz}$ *is the greatest modal companion of* $\mathsf{M}^+\mathsf{IPC}$.

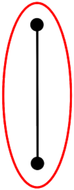
Two main ingredients:

Theorem

M^+Grz *is the greatest modal companion of* M^+IPC.

Two main ingredients:

- A finite MIPC-frame validates MCas iff every E-class is clean, i.e. if xEy and xRy, then $x = y$.



Theorem

$\mathsf{M^+Grz}$ *is the greatest modal companion of* $\mathsf{M^+IPC}$.

Two main ingredients:

- A finite MIPC-frame validates MCas iff every E-class is clean, i.e. if xEy and xRy, then $x = y$.
- $\mathsf{M^+Grz}$ has the finite model property.

Conclusion

Summary

Negative result

There is no greatest modal companion of MIPC.

Summary

Negative result

There is no greatest modal companion of MIPC.

Positive result

$M^{+}Grz$ is the greatest modal companion of $M^{+}IPC$.

Open problems and future directions

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Does every extension of MIPC have a modal companion?

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Does every extension of MIPC have a modal companion?

If not, does every extension of M^+IPC have a modal companion?

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Does Blok-Esakia hold in the monadic setting? That is, is σ a lattice isomorphism between the lattice of extensions of MIPC and of MGrz?

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Does every extension of MIPC have a modal companion?
If not, does every extension of M^+IPC have a modal companion?

Open problem

Does Blok-Esakia hold in the monadic setting? That is, is σ a lattice isomorphism between the lattice of extensions of MIPC and of MGrz?

Conjecture: no.
If it is not, does Blok-Esakia hold for the extensions of M^+IPC and M^+Grz?

THANK YOU!