Bitopological duality for some subordination Boolean algebras including compingent and De Vries algebras

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BLAST 2021, New Mexico

Introducction

The aim of this talk is to present a bitopological representation for some classes of Boolean algebras with a subordination relation.

1. Bezhanishvili G, Morandi R, Patrick J.: *Topo-canonical completions of closure algebras and Heyting algebras,* Algebra univers. Vol. 58 (1), 2008, 1-34.
2. Bezhanishvili G, Bezhanishvili N., Gabelaia D, Kurz A.: *Bitopological duality for distributive lattices and Heyting algebras.* Mathematical Structures in Computer Science. 20(3), 359-93 (2010).
3. A. Jung and M. A. Moshier. On the bitopological nature of Stone duality. Technical Report CSR-06-13, School of Computer Science, University of Birmingham, 2006.

The bitopological spaces that we are going to present are essentially Stone spaces (X, τ) with an additional topology τ_S. This new topology is generated by a special basis S.

Subordinations relations

A subordination on a Boolean algebra A is a binary relation

$$\prec \subseteq A \times A$$

satisfying the following conditions:

(S1) $0 \prec 0$ and $1 \prec 1$,

(S2) $a \prec b, c \Rightarrow a \prec b \wedge c$,

(S3) $a, b \prec c \Rightarrow a \vee b \prec c$,

(S4) $a \leq b \prec c \leq d \Rightarrow a \prec d$.

Also, it is easy to see that a binary relation $\prec$ on A is a subordination iff

$$\downarrow a = \{x \in A : x \prec a\} \text{ is an ideal, and } \uparrow a = \{y \in A : a \prec y\} \text{ is a filter,}$$

for each $a \in A$.

A Boolean algebra with a subordination is called a **subordination algebra**

Equivalent presentations

Subordination can be defined in two equivalent ways. As **precontat** relations or as **quasi-modal operators**.

A **precontact** relation on a Boolean algebra A is a binary relation δ satisfying

(P1) $a\delta b \Rightarrow a, b \neq 0$,

(P2) $a\delta b \vee c \Leftrightarrow a\delta b$ or $a\delta c$,

(P3) $s \vee b\delta c \Leftrightarrow a\delta b$ or $b\delta c$.

- Dimov G. and Vakarelov D.*:* Topological representation of precontact algebras, in Lecture Notes Comp. Sci., 3929, W. MacCaull et al. (eds.), Springer-Verlag, Berlin (2006), pp. 1-16.
- Düntsch, I., Vakarelov, D.: Region-based theory of discrete spaces: a proximity approach. Ann. Math. Artif. Intell. 49(1–4), 5–14 (2007)

Precontact and subordination relations are duals notions

$$\prec \longrightarrow \delta_{\prec} = \{(a,b) \in A \times A : a \not\prec \neg b\}$$

and

$$\delta \longrightarrow \prec_{\delta} = \{(a,b) \in A \times A : a \not\delta \neg b\}.$$

A quasi-modal operator on a Boolean algebra A is a function

$$\Delta : A \to \mathrm{Id}(A)$$

satisfying the following conditions:

(Q1) $\Delta 1 = A$.

(Q2) $\Delta(a \wedge b) = \Delta a \cap \Delta b$, for all $a, b \in A$.

A quasi-modal algebra is a pair (A, Δ) where A is a Boolean algebra and Δ is a quasi-modal operator on A.

- Celani, S.: Quasi-modal algebras. Math. Bohem. 126(4), 721–736 (2001)

$$\prec \quad \longrightarrow \quad \Delta_{\prec} a = \downarrow a = \{y \in A : x \prec a\}$$

$$\Delta \quad \longrightarrow \quad \prec_{\Delta} = \{(a, b) : a \in \Delta b\}$$

Thus, we can say that in Boolean algebras the notions of **subordination relations, precontat relations and quasi-modal operators are equivalent notions.**

Classes of subordination algebras

- **A quasi-topological algebra,** or $S4$-**subordination algebra**, is a subordination algebra $\langle A, \prec \rangle$ satisfying the additional conditions
  (S5) $a \prec b \Rightarrow a \leq d$.
  (S6) $a \prec b \Rightarrow$ there exists $c \in A$ $(a \prec c \prec b)$.
- **A quasi-monadic algebra** or $S5$-**subordination algebra** is a quasi-topological algebra $\langle A, \prec \rangle$ satisfying the additional condition:
  (S7) $a \prec b \Rightarrow \neg b \prec \neg a$.
- A **compingent algebra** is a quasi-monadic algebra $\langle A, \prec \rangle$ such that
  (S8) $a \neq 0 \Rightarrow \exists b \neq 0$ $(b \prec a)$.
- A **de Vries algebra** is a **complete** compingent algebra.

Subordinations and modal operators

Modal operators in Boolean algebras are connected with subordinations relations

- A **modal** operator on a Boolean algebra A is a function $\Box : A \to A$ satisfying the conditions:

$$\Box 1 = 1 \text{ and } \Box(a \wedge b) = \Box a \wedge \Box b.$$

  A **modal algebra** is a Boolean algebra A with a modal operator $\Box$.

- **A topological** algebra is a modal algebra $\langle A, \Box \rangle$ satisfying the equations:

$$\Box a \leq a \text{ and } \Box a \leq \Box\Box a.$$

- **A monadic** algebra is a topological algebra satisfying the additional equation:

$$\Box\,(a \vee \Box b) = \Box a \vee \Box b.$$

Every modal operator $\Box : A \to A$ define a subordination

$$\prec_{\Box} \subseteq A \times A$$

in the following way:

$$a \prec_{\Box} b \text{ iff } a \leq \Box b.$$

It is easy to check that

$$\langle A, \Box \rangle \text{ is a topological algebra iff } \langle A, \prec_{\Box} \rangle \text{ is a } S4\text{-subordination algebra.}$$

$$\langle A, \Box \rangle \text{ is a monadic algebra iff } \langle A, \prec_{\Box} \rangle \text{ is a } S5\text{-subordination algebra.}$$

Therefore, the theory of subordination algebras is strongly connected with the theory of modal algebras.

Stone space of a Boolean algebra

Recall that the **Stone space** of a Boolean algebra A is a topological space

$$\langle X, \tau \rangle$$

where X is the set of prime filters of A, and τ is a topology generated by the family $\{\varphi(a) : a \in A\}$, where $\varphi : A \to \mathcal{P}(X)$ defined by

$$\varphi(a) = \{x \in X : a \in x\}.$$

$$\text{Ideals of } A \quad \longleftrightarrow \quad \text{open subsets of } \langle X, \tau \rangle$$

$$I \quad \longleftrightarrow \quad \beta(I) = \{x \in X : I \cap x \neq \emptyset\}$$

In particular, if $\prec$ is a subordination defined on A, then we have

$$\downarrow a = \{b \in A : b \prec a\} \longrightarrow \beta(\downarrow a) = \{x \in X : \downarrow a \cap x \neq \emptyset\}.$$

Bitopological spaces of a S4-subordination algebra

Theorem

Let $\langle A, \prec \rangle$ be a S4-subordination algebra and let $\langle X, \tau \rangle$ be the Stone space of A. Then

1. *The family $S = \{\beta(\downarrow a) : a \in A\}$ is closed under finite intersections and is a basis for a topology τ_S on X;*
2. *Every open of τ_S is an open in the Stone topology, i.e. $\tau_S \subseteq \tau$;*
3. *For each $a \in A$*
$$\mathrm{int}_S(\varphi(a)) = \beta(\downarrow a).$$
4. *The Stone map $\varphi : A \to \mathrm{Clop}(X)$ satisfies the property*
$$a \prec b \iff \varphi(a) \prec_S \varphi(b)$$
where the relation $\prec_S \subseteq \mathrm{Clop}(X) \times \mathrm{Clop}(X)$ is the relation given by
$$U \prec_S V \text{ iff } U \subseteq \mathrm{int}_S(V).$$

Definition

*A bitopological **subordination** space is a bitopological space*

$$\langle X, \tau, \tau_S \rangle ,$$

1. $\langle X, \tau \rangle$ *is a Stone space.*
2. *The family*
$$S = \{\mathrm{int}_S(U) : U \in \mathrm{Clop}(X, \tau)\}$$
is closed under finite intersections and is a basis for the topology τ_S.
3. *For each* $U \in \mathrm{Clop}(X)$,
$$\mathrm{int}_S(U) \in \tau.$$
4. *The relation* $\prec_S \subseteq \mathrm{Clop}(X) \times \mathrm{Clop}(X)$ *defined by*
$$U \prec_S V \text{ iff } U \subseteq \mathrm{int}_S(V),$$
satisfies the property
$$U \prec_S V \Rightarrow \exists W \in \mathrm{Clop}(X) \ (U \prec_S W \text{ and } W \prec_S V) .$$

Let (X, τ, τ_S) be a bitopological subordination space.

Since the family

$$S = \{\mathrm{int}_S(U) : U \in \mathrm{Clop}(X)\}$$

is a basis of the topology τ_S, the specialization or topological **preorder** $\leq_S$ of (X, τ_S) can be defined as:

$$x \leq_S y \Longleftrightarrow \forall U \in \mathrm{Clop}(X)\ \ (x \in \mathrm{int}_S(U) \Longrightarrow y \in \mathrm{int}_S(U)).$$

Consider a new relation $\lhd_S \subseteq X \times X$

$$x \lhd_S y \overset{def}{\Longleftrightarrow} \forall U \in \mathrm{Clop}(X)\ \ (x \in \mathrm{int}_S(U) \Longrightarrow y \in U).$$

Always

$$\leq_S \subseteq \lhd_S.$$

We can prove the condition

$$U \prec_S V \Rightarrow \exists W \in \mathrm{Clop}(X)\ (U \prec_S W \prec_S V)\,.$$

is equivalent to the condition

$$\lhd_S \subseteq \leq_S\ .$$

Thus, the definition of topological subordination space can be given in following equivalent way

Definition

A bitopological ***subordination*** *space is a bitopological space*

$$\langle X, \tau, \tau_S \rangle ,$$

such that

1. $\langle X, \tau \rangle$ *is a Stone space.*
2. *The family*
$$S = \{\mathrm{int}_S(U) : U \in \mathrm{Clop}(X)\}$$
is closed under finite intersections and is a basis for the topology τ_S.
3. *For each* $U \in \mathrm{Clop}(X)$,
$$\mathrm{int}_S(U) \in \tau.$$
4. $\lhd_S \subseteq \leq_S$.

Theorem

If $\langle X, \tau, \tau_S \rangle$ *is a bitopological subordination space, then*

$$\langle \mathrm{Clop}(X), \prec_S \rangle$$

is a $S4$*-subordination algebra, where the relation* $\prec_S \subseteq \mathrm{Clop}(X) \times \mathrm{Clop}(X)$ *is given by*

$$U \prec_S V \text{ is } U \subseteq \mathrm{int}_S(V).$$

Theorem

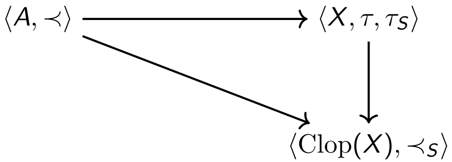
If $\langle A, \prec \rangle$ *is an* $S4$*-subordination algebra and* $\langle X, \tau \rangle$ *is the Stone space of* A*, then*

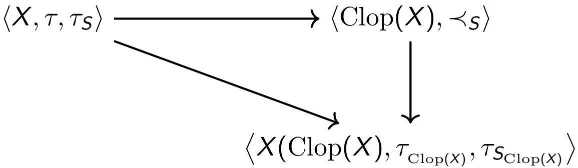
$$\langle X, \tau, \tau_S \rangle$$

is a bitopological subordination space and the Stone map $\varphi : A \to \mathrm{Clop}(X)$ *is a Boolean isomorphism satisfying the following condition:*

$$a \prec b \text{ iff } \varphi(a) \prec_S \varphi(b).$$

Synthesis


$\langle A, \prec \rangle$
$\langle X, \tau, \tau_S \rangle$
$\langle \mathrm{Clop}(X), \prec_S \rangle$


$\langle X, \tau, \tau_S \rangle$
$\langle \mathrm{Clop}(X), \prec_S \rangle$
$\left\langle X(\mathrm{Clop}(X), \tau_{\mathrm{Clop}(X)}, \tau_{S_{\mathrm{Clop}(X)}} \right\rangle$

Dual spaces for Comp and DeV

$$\text{Comp} = \text{SubS4} + \{a \prec b \Rightarrow \neg b \prec \neg a\} + \{a \neq 0 \Rightarrow \exists b \neq 0 \ (b \prec a)\}$$

DeV = **complete** compingent algebras.

Dual spaces for Comp

- Bitopological subordination spaces $\langle X, \tau, \tau_S \rangle$ such that
  1. The preorder $\lhd_S$ is an equivalence relation, and
  2. $\mathrm{int}_S(U) \neq \emptyset$, for each $U \in \mathrm{Clop}(X) - \{\emptyset\}$ (compingent bitopological spaces).

Recall that the relation $\lhd_S$ is given by

$$\begin{aligned} \lhd_S &= \{(x, y) : \forall U \in \mathrm{Clop}(X) \ (x \in \mathrm{int}_S(U) \Rightarrow y \in U)\}. \\ &= \{(x, y) : \forall U \in \mathrm{Clop}(X) \ (x \in \mathrm{int}_S(U) \Rightarrow y \in \mathrm{int}_S(U))\}. \end{aligned}$$

Dual spaces for DeV

- Compingent bitopological spaces, $\langle X, \tau, \tau_S \rangle$ such that the Stone space $\langle X, \tau \rangle$ is extremally disconnected.
  We call these bitopological spaces de *Vries spaces.*

Categorical dualities

Now, we have only a bitopological representation, but we need to have categorical dualities.

Question:

What is the correct notion of morphism for subordination algebras?

Answer:

It depends on the motivation

We can take

meet-homomorphisms + conditions for the subordination relations

or

Boolean homomorphisms + conditions for the subordination relations

Morphisms based in meet-homomorphisms

Let $h : A_1 \to A_2$ be a meet-homomorphism between subordination algebras and consider the following possible conditions to define morphisms:

(1) $a \prec b$ implies $h(a) \prec h(b)$

(2) $a \prec h(b) \Rightarrow \exists c\,(a \leq h(c)$ and $c \prec b)$

(3) $a \prec b \Rightarrow \neg h(\neg a) \prec h(b)$ this condition implies (1)

With the above conditions we can define different classes of morphisms between $S4$-subordination algebras.
We can take some of the following combinations to define morphisms

$$(1)\,,\ \text{or}\quad (3)\,,\ \text{or}\ \ (1)+(2)\,,\quad (2)+(3)$$

$$(1)\ a \prec b \text{ implies } h(a) \prec h(b)$$

$$(2)\ a \prec h(b) \Rightarrow \exists c\,(a \leq h(c) \text{ and } c \prec b)$$

The conditions (1) and (2) are a generalization of the notion of modal homomorphism. In effect:

$$\downarrow a = \Delta a = \{b : b \prec a\}$$

$a \prec b$ implies $h(a) \prec h(b)$	$\equiv$	$h[\Delta a] \subseteq \Delta h(a)$	$h(\Box a) \leq \Box h(a)$
$a \prec h(b) \Rightarrow \exists c\,(a \leq h(c)$ and $c \prec b)$	$\equiv$	$\Delta h(a) \subseteq (h[\Delta a]]$	$\Box h(a) \leq h(\Box a)$

Morphisms based in meet-homomorphisms

Meet-homomorphisms	**Boolean relations**
$h : A_1 \to A_2$	$R \subseteq X_1 \times X_2$
$h(1) = 1$ $h(a \wedge b) = h(a) \wedge h(b)$	$R(x)$ closed subset of X_2, $h_R(U) = \{x : R(x) \subseteq U\} \in \mathrm{Clop}(X_1)$

$$(x, y) \in R_h \text{ sii } h^{-1}(x) \subseteq y$$

A **round filter** of $S4$-subordination algebra $\langle A, \prec \rangle$ is a filter F satisfying the following condition:

$$a \in F \Longrightarrow \exists b \in F \ (b \prec a) .$$

Topological characterizations

Theorem

Let $h : A_1 \to A_2$ be a meet-homomorphism between two S4-subordination algebras. Let $\langle X_1, \tau_1, \tau_{S_1} \rangle$ and $\langle X_2, \tau_2, \tau_{S_2} \rangle$ be the bitopological subordination spaces. Let $R = R_h$ be the Boolean relation defined by h.

1. *The following conditions are equivalent:*
   (a) $a \prec b$ *implies* $h(a) \prec h(b)$.
   (b) $h_R(\mathrm{int}_{S_1}(U)) \subseteq \mathrm{int}_{S_2}(h_R(U))$, *for each* $U \in \mathrm{Clop}(X_1)$.
2. *The following conditions are equivalent:*
   (a) $a \prec h(b) \Rightarrow \exists c\,(a \leq h(c)$ *and* $c \prec b)$.
   (b) $h^{-1}(F)$ *is a round filter of* A_1, *for each round filter* F *of* A_2,
   (c) $\mathrm{int}_{S_2}(h_R(U)) \subseteq h_R(\mathrm{int}_{S_1}(U))$, *for all* $U \in \mathrm{Clop}(X_1)$.
3. *The following conditions are equivalent:*
   (a) $a \prec b \Rightarrow \neg h(\neg a) \prec h(b)$, *for all* $a, b \in A_1$.
   (b) $g_R(\mathrm{int}_{S_1}(U)) \subseteq \mathrm{int}_{S_2}(h_R(U))$, *for all* $U \in \mathrm{Clop}(X_1)$, *where* $g_R(U) = h_R(U^c)^c$.

We recall that a map $h : A \to B$ is a Boolean homomorphism iff

1. $h^{-1}(F)$ of A_1 is a filter for each filter F of F_2.
2. $h^{-1}(P)$ is a maximal filter of A_1 for each maximal filter of A_2.

In the context of $S5$-subordination algebras we have a similar result.

$$\mathrm{SubS5} = \mathrm{SubS4} + \{a \prec b \Rightarrow \neg b \prec \neg a\}.$$

Theorem

Let $h : A_1 \to A_2$ be a meet-homomorphism between two S5-subordination algebras. Then the following conditions are equivalent:

1. *h satisfies the following conditions:*
    1. $a \prec b \Rightarrow \neg h(\neg a) \prec h(b)$, *for all* $a, b \in A_1$.
    2. $a \prec h(b) \Rightarrow \exists c\,(a \leq h(c)$ *and* $c \prec b)$.
2. *h satisfies the following conditions:*
    1. $h^{-1}(F)$ *is a round filter of* A_1 *for each round filter* F *of* A_2.
    2. $h^{-1}(P)$ *is a maximal round filter of* A_1 *for each maximal round filter* P *of* A_2.

Categorical dualities

Thus, we can define different categories taking the same objects but with different morphisms.

SubS4

- **Objects:** $S4$-subordination algebras
- **Morphisms:** Meet-homomorphisms $h : A_1 \to A_2$ satisfying the conditions:
  - $a \prec b$ implies $h(a) \prec h(b)$,
  - $a \prec h(b) \Rightarrow \exists c\,(a \leq h(c)$ and $c \prec b)$.

BitoSub

- **Objects:** bitopological subordination spaces
- **Morphisms:** Boolean relations $R \subseteq X_1 \times X_2$ satisfying the condition:
  - $\mathrm{int}_{S_1}(h_R(U)) = h_R(\mathrm{int}_{S_2}(U))$, for every $U \in \mathrm{Clop}(X_2)$.

Theorem

The categories **SubS4** *and* **BitoSub** *are dually equivalent.*

The category **SubS4** is related to the category of **quasi-topological algebras** introduced in Celani S. A.: Quasi-Modal algebras, Mathematica Bohemica Vol. 126, No. 4 (2001),.

Bitopological duality for de Vries algebras

DeV

- **Objects:** de Vries algebras
- **Morphisms:** de Vries homomorphisms, i.e., meet-homomorphisms $h : A_1 \to A_2$ satisfying the conditions:
  - $a \prec b$ implies $\neg h(\neg a) \prec h(b)$,
  - $h(a) = \bigvee \{h(b) : b \prec a\}$.

BitoDeV

- **Objects:** Bitopological de Vries spaces, i.e., bitopological subordination spaces $\langle X, \tau, \tau_S \rangle$ satisfying the conditions:
  - $\langle X, \tau \rangle$ is extremally disconnected.
  - $\lhd_S$ is an equivalence relation.
  - $\mathrm{int}_S(U) \neq \emptyset$, para cada $U \in \mathrm{Clop}(X) - \{\emptyset\}$
- **Morphisms:** de Vries relations, i.e., Boolean relations $R \subseteq X_1 \times X_2$ satisfying the condition:
  - $R(x) \neq \emptyset$, for each $x \in X_1$,
  - $g_R(\mathrm{int}_{S_2}(U)) \subseteq \mathrm{int}_{S_1}(h_R(U))$ for every $U \in \mathrm{Clop}(X_2)$.
  - $h_R(U) = \mathrm{cl}(h_R(\mathrm{int}_{S_1}(U)))$, for every $U \in \mathrm{Clop}(X_2)$.

Theorem

The categories **Dev** *and* **BitoDeV** *are dually equivalent.*

We note that the category **DeV** is the category of *complete compingent algebras* study by de Vries in H.: Compact spaces and compactifications. An algebraic approach, PhD thesis, University of Amsterdam, 1962

Remarks

1. The usual composition of de Vries homomorphism is not a de Vries homomorphism, and
2. the usual composition of de Vries relations is not a de Vries relation.

Thus, we need to define new compositions.

- The definition of the proper composition between de Vries homomorphism is given by de Vries

De Vries, H.: Compact spaces and compactifications. An algebraic approach, PhD thesis, University of Amsterdam, 1962.

- The definition of the proper composition between de Vries relations is given in

Bezhanishvili, G., Bezhanishvili, N., Sourabh, S., and Venema, Y.: Irreducible equivalence relations, Gleason spaces, and de Vries duality, Appl. Categ. Structures, 2016.

An application. Subordination congruences

Definition

Let $\langle A, \prec \rangle$ be a subordination algebra. A Boolean congruence $\theta \subseteq A \times A$ is a ***subordination congruence*, or *quasi-modal congruence*,** if θ satisfies the following condition.

If $c \prec a$ and $(a, b) \in \theta$, then there exists $d \in A$ such that $d \prec b$ and $(c, d) \in \theta$.

$$\prec \circ \theta \subseteq \theta \circ \prec .$$

We can prove that

- The lattice of subordination congruences is isomorphic to the lattice of round filters,
- Any subordination congruence is the kernel of a Boolean homomorphism $h : A \to B$ satisfying the conditions:
  1. $a \prec b$ implies $h(a) \prec h(b)$,
  2. $a \prec h(b) \Rightarrow \exists c\,(a \leq h(c)$ and $c \prec b)$.

We recall that any closed subset Y of a Stone space $\langle X, \tau \rangle$ define a Boolean congruence

$$\theta(Y) = \{(a, b) \in A \times A : \varphi(a) \cap Y = \varphi(b) \cap Y\}.$$

A subset $Y \subseteq X$ is **saturated** of $\langle X, \tau_S \rangle$ if

$$Y = \bigcap \{O \in \tau_S : Y \subseteq O\}.$$

Theorem

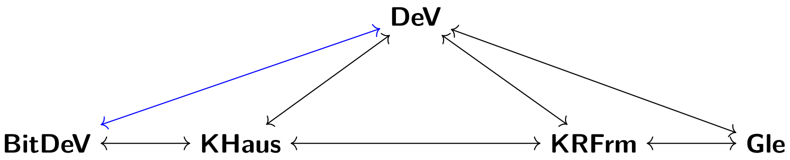
The lattice of subordination congruences of a S4-subordination algebra $\langle A, \prec \rangle$ is isomorphic to the lattice of subsets which are closed in the Stone topology τ and saturated in the topology τ_S:

$$\mathrm{Con}\,\langle A, \prec \rangle \cong \mathcal{C}\,\langle X, \tau \rangle \cap \mathcal{S}\,\langle X, \tau_S \rangle\,.$$

Conclusion

In this talk we have shown that there exists bitopological dualities for some classes of Boolean algebras with a subordination.
In particular, we have shown a new duality for de Vries algebras.

1. **DeV**: Category of de Vries algebras



1. **BitDeV**: Bitopological de Vries spaces;
2. **KHaus** : Category of compact and Hausdorff spaces (de Vries duality)
3. **KRFrm**: Category of compact regular frames (Isbell theorem).
4. **Gle** : Category of Gleason spaces (Stone spaces with an equivalence relation satisfying certains conditions)

For future work

(1) We want to give a direct proof of the dual equivalence

$$\textbf{BitoDeV} \cong^d \textbf{KRFrm}.$$

(2) The definition of subordination relation does not use Boolean negation. So, we can study bitopological dualities to bounded distributive lattices and Heyting algebras with a S4-subordination relation.

(3) A common generalization of de Vries algebras and stably compact frames is the class PrFrm of *proximity frames.*
This class is studied in
Guram Bezhanishvili and John Harding, *Proximity Frames and Regularization,* Appl Categor Struct (2014) 22:43–78.

We suggest developing a bitopological duality for the class of proximity frames.

Thanks

Muchas Gracias

Please, for more details write to sergiocelani@gmail.com. I will be very happy to answer