Bitopological duality for some subordination Boolean algebras including compingent and De Vries algebras

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The aim of this talk is to present a bitopological representation for some classes of Boolean algebras with a subordination relation.


The bitopological spaces that we are going to present are essentially Stone spaces \((X, \tau)\) with an additional topology \(\tau_S\). This new topology is generated by a special basis \(S\).
Subordinations relations

A subordination on a Boolean algebra $A$ is a binary relation

$$
\prec \subseteq A \times A
$$

satisfying the following conditions:

(S1) $0 \prec 0$ and $1 \prec 1$,

(S2) $a \prec b, c \Rightarrow a \prec b \land c$,

(S3) $a, b \prec c \Rightarrow a \lor b \prec c$,

(S4) $a \leq b \prec c \leq d \Rightarrow a \prec d$.

Also, it is easy to see that a binary relation $\prec$ on $A$ is a subordination iff

$$
\downarrow a = \{ x \in A : x \prec a \} \text{ is an ideal, and } \uparrow a = \{ y \in A : a \prec y \} \text{ is a filter,}
$$

for each $a \in A$.

A Boolean algebra with a subordination is called a subordination algebra
Equivalent presentations

Subordination can be defined in two equivalent ways. As precontat relations or as quasi-modal operators.

A precontact relation on a Boolean algebra $A$ is a binary relation $\delta$ satisfying

(P1) $a \delta b \Rightarrow a, b \neq 0$,

(P2) $a \delta b \lor c \iff a \delta b \text{ or } a \delta c$,

(P3) $s \lor b \delta c \iff a \delta b \text{ or } b \delta c$.


Precontact and subordination relations are duals notions

$$\prec \quad \longrightarrow \quad \delta \prec = \{(a, b) \in A \times A : a \not\preceq \neg b\}$$

and

$$\delta \quad \longrightarrow \quad \prec_\delta = \{(a, b) \in A \times A : a \not\delta \neg b\}.$$
A quasi-modal operator on a Boolean algebra $A$ is a function

$$\Delta : A \to \text{Id}(A)$$

satisfying the following conditions:

(Q1) $\Delta 1 = A$.

(Q2) $\Delta (a \land b) = \Delta a \cap \Delta b$, for all $a, b \in A$.

A quasi-modal algebra is a pair $(A, \Delta)$ where $A$ is a Boolean algebra and $\Delta$ is a quasi-modal operator on $A$.


$\prec \quad \rightarrow \quad \Delta \prec a = \downarrow a = \{y \in A : x \prec a\}$

$\Delta \quad \rightarrow \quad \prec \Delta = \{(a, b) : a \in \Delta b\}$
Thus, we can say that in Boolean algebras the notions of subordination relations, precontat relations and quasi-modal operators are equivalent notions.
Classes of subordination algebras

- **A quasi-topological algebra**, or *S4-subordination algebra*, is a subordination algebra \( \langle A, \prec \rangle \) satisfying the additional conditions
  - (S5) \( a \prec b \Rightarrow a \leq d \).
  - (S6) \( a \prec b \Rightarrow \) there exists \( c \in A \) (\( a \prec c \prec b \)).

- **A quasi-monadic algebra** or *S5-subordination algebra* is a quasi-topological algebra \( \langle A, \prec \rangle \) satisfying the additional condition:
  - (S7) \( a \prec b \Rightarrow \neg b \prec \neg a \).

- **A compingent algebra** is a quasi-monadic algebra \( \langle A, \prec \rangle \) such that
  - (S8) \( a \neq 0 \Rightarrow \exists b \neq 0 \) (\( b \prec a \)).

- **A de Vries algebra** is a complete compingent algebra.
Subordinations and modal operators

Modal operators in Boolean algebras are connected with subordinations relations

- A **modal** operator on a Boolean algebra $A$ is a function $\Box : A \rightarrow A$ satisfying the conditions:
  \[
  \Box 1 = 1 \text{ and } \Box(a \land b) = \Box a \land \Box b.
  \]

  A **modal algebra** is a Boolean algebra $A$ with a modal operator $\Box$.

- A **topological** algebra is a modal algebra $\langle A, \Box \rangle$ satisfying the equations:
  \[
  \Box a \leq a \text{ and } \Box a \leq \Box \Box a.
  \]

  A **monadic** algebra is a topological algebra satisfying the additional equation:
  \[
  \Box (a \lor \Box b) = \Box a \lor \Box b.
  \]
Every modal operator $\Box : A \rightarrow A$ define a subordination

$\prec_\Box \subseteq A \times A$

in the following way:

$a \prec_\Box b$ iff $a \leq \Box b$.

It is easy to check that

$\langle A, \Box \rangle$ is a topological algebra iff $\langle A, \prec_\Box \rangle$ is a S4-subordination algebra.

$\langle A, \Box \rangle$ is a monadic algebra iff $\langle A, \prec_\Box \rangle$ is a S5-subordination algebra.

Therefore, the theory of subordination algebras is strongly connected with the theory of modal algebras.
Stone space of a Boolean algebra

Recall that the **Stone space** of a Boolean algebra $A$ is a topological space 

$$\langle X, \tau \rangle$$

where $X$ is the set of prime filters of $A$, and $\tau$ is a topology generated by the family $\{\varphi(a) : a \in A\}$, where $\varphi : A \to \mathcal{P}(X)$ defined by 

$$\varphi(a) = \{x \in X : a \in x\}.$$ 

Ideals of $A$ $\longleftrightarrow$ open subsets of $\langle X, \tau \rangle$

$I$ $\longleftrightarrow$ $\beta(I) = \{x \in X : I \cap x \neq \emptyset\}$

In particular, if $\prec$ is a subordination defined on $A$, then we have

$$\downarrow a = \{b \in A : b \prec a\} \longrightarrow \beta(\downarrow a) = \{x \in X : \downarrow a \cap x \neq \emptyset\}.$$
Theorem

Let \( \langle A, \prec \rangle \) be a S4-subordination algebra and let \( \langle X, \tau \rangle \) be the Stone space of A. Then

1. The family \( S = \{ \beta(\downarrow a) : a \in A \} \) is closed under finite intersections and is a basis for a topology \( \tau_S \) on \( X \);

2. Every open of \( \tau_S \) is an open in the Stone topology, i.e. \( \tau_S \subseteq \tau \);

3. For each \( a \in A \)
   \[ \text{int}_S(\varphi(a)) = \beta(\downarrow a). \]

4. The Stone map \( \varphi : A \to \text{Clop}(X) \) satisfies the property
   \[ a \prec b \iff \varphi(a) \prec_S \varphi(b) \]
   where the relation \( \prec_S \subseteq \text{Clop}(X) \times \text{Clop}(X) \) is the relation given by
   \[ U \prec_S V \text{ iff } U \subseteq \text{int}_S(V). \]
Definition

A bitopological **subordination** space is a bitopological space

\[ \langle X, \tau, \tau_S \rangle, \]

1. \( \langle X, \tau \rangle \) is a Stone space.
2. The family
   \[ S = \{ \text{int}_S(U) : U \in \text{Clop}(X, \tau) \} \]
   is closed under finite intersections and is a basis for the topology \( \tau_S \).
3. For each \( U \in \text{Clop}(X) \),
   \[ \text{int}_S(U) \in \tau. \]
4. The relation \( \prec_S \subseteq \text{Clop}(X) \times \text{Clop}(X) \) defined by
   \[ U \prec_S V \text{ iff } U \subseteq \text{int}_S(V), \]
   satisfies the property
   \[ U \prec_S V \Rightarrow \exists W \in \text{Clop}(X) \ (U \prec_S W \text{ and } W \prec_S V). \]
Let \((X, \tau, \tau_S)\) be a bitopological subordination space.

Since the family
\[ S = \{ \text{int}_S(U) : U \in \text{Clop}(X) \} \]
is a basis of the topology \(\tau_S\), the specialization or topological preorder \(\leq_S\) of \((X, \tau_S)\) can be defined as:
\[ x \leq_S y \iff \forall U \in \text{Clop}(X) (x \in \text{int}_S(U) \implies y \in \text{int}_S(U)). \]

Consider a new relation \(\triangleleft_S \subseteq X \times X\)
\[ x \triangleleft_S y \overset{\text{def}}{\iff} \forall U \in \text{Clop}(X) (x \in \text{int}_S(U) \implies y \in U). \]

Always
\[ \leq_S \subseteq \triangleleft_S. \]

We can prove the condition
\[ U \triangleleft_S V \implies \exists W \in \text{Clop}(X) (U \triangleleft_S W \triangleleft_S V). \]
is equivalent to the condition
\[ \triangleleft_S \subseteq \leq_S. \]
Thus, the definition of topological subordination space can be given in following equivalent way

**Definition**

A bitopological **subordination** space is a bitopological space

\[ \langle X, \tau, \tau_S \rangle, \]

such that

1. \( \langle X, \tau \rangle \) is a Stone space.
2. The family

\[ S = \{ \text{int}_S(U) : U \in \text{Clop}(X) \} \]

is closed under finite intersections and is a basis for the topology \( \tau_S \).
3. For each \( U \in \text{Clop}(X) \),

\[ \text{int}_S(U) \in \tau. \]
4. \( S \subseteq \leq S \).

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Theorem

If $\langle X, \tau, \tau_S \rangle$ is a bitopological subordination space, then

$$\langle \text{Clop}(X), \prec_S \rangle$$

is a $S_4$-subordination algebra, where the relation $\prec_S \subseteq \text{Clop}(X) \times \text{Clop}(X)$ is given by

$$U \prec_S V \text{ is } U \subseteq \text{int}_S(V).$$

Theorem

If $\langle A, \prec \rangle$ is an $S_4$-subordination algebra and $\langle X, \tau \rangle$ is the Stone space of $A$, then

$$\langle X, \tau, \tau_S \rangle$$

is a bitopological subordination space and the Stone map $\varphi : A \rightarrow \text{Clop}(X)$ is a Boolean isomorphism satisfying the following condition:

$$a \prec b \iff \varphi(a) \prec_S \varphi(b).$$
Synthesis

\[ \langle A, \prec \rangle \rightarrow \langle X, \tau, \tau_S \rangle \]

\[ \langle \text{Clop}(X), \prec \rangle \]

\[ \langle X, \tau, \tau_S \rangle \rightarrow \langle \text{Clop}(X), \prec \rangle \]

\[ \langle X(\text{Clop}(X)), \tau_{\text{Clop}(X)}, \tau_{S_{\text{Clop}(X)}} \rangle \]
Dual spaces for Comp and DeV

\[
\text{Comp} = \text{SubS4} + \{a \prec b \Rightarrow \neg b \prec \neg a\} + \{a \neq 0 \Rightarrow \exists b \neq 0 (b \prec a)\}
\]

\[
\text{DeV} = \text{complete} \text{ compingent algebras.}
\]

Dual spaces for Comp

- Bitopological subordination spaces \(\langle X, \tau, \tau_S \rangle\) such that
  1. The preorder \(\triangleleft_S\) is an equivalence relation, and
  2. \(\text{int}_S(U) \neq \emptyset\), for each \(U \in \text{Clop}(X) - \{\emptyset\}\) (compingent bitopological spaces).

Recall that the relation \(\triangleleft_S\) is given by

\[
\triangleleft_S = \{(x, y) : \forall U \in \text{Clop}(X) \ (x \in \text{int}_S(U) \Rightarrow y \in U)\}.
\]

Dual spaces for DeV

- Compingent bitopological spaces, \(\langle X, \tau, \tau_S \rangle\) such that the Stone space \(\langle X, \tau \rangle\) is extremally disconnected.
  We call these bitopological spaces de Vries spaces.
Categorical dualities

Now, we have only a bitopological representation, but we need to have categorical dualities.

Question:

What is the correct notion of morphism for subordination algebras?

Answer:

It depends on the motivation

We can take

meet-homomorphisms + conditions for the subordination relations

or

Boolean homomorphisms + conditions for the subordination relations
Morphisms based in meet-homomorphisms

Let \( h : A_1 \to A_2 \) be a meet-homomorphism between subordination algebras and consider the following possible conditions to define morphisms:

1. \( a \prec b \) implies \( h(a) \prec h(b) \)

2. \( a \prec h(b) \Rightarrow \exists c \ (a \leq h(c) \text{ and } c \prec b) \)

3. \( a \prec b \Rightarrow \neg h(\neg a) \prec h(b) \) \hspace{.5cm} \text{this condition implies (1)}

With the above conditions we can define different classes of morphisms between \( S_4\)-subordination algebras.

We can take some of the following combinations to define morphisms

\( (1), \text{ or } (3), \text{ or } (1) + (2), \text{ or } (2) + (3) \)
(1) $a \prec b$ implies $h(a) \prec h(b)$

(2) $a \prec h(b) \Rightarrow \exists c \ (a \leq h(c) \text{ and } c \prec b)$

The conditions (1) and (2) are a generalization of the notion of modal homomorphism. In effect:

$$
\downarrow a = \Delta a = \{ b : b \prec a \}
$$

\[
\begin{align*}
    a \prec b \text{ implies } h(a) \prec h(b) & \equiv h[\Delta a] \subseteq \Delta h(a) & h(\square a) \leq \square h(a) \\
    a \prec h(b) \Rightarrow \exists c \ (a \leq h(c) \text{ and } c \prec b) & \equiv \Delta h(a) \subseteq (h[\Delta a]) & \square h(a) \leq h(\square a)
\end{align*}
\]
Morphisms based in meet-homomorphisms

Meet-homomorphisms

\[ h : A_1 \rightarrow A_2 \]
\[ h(1) = 1 \]
\[ h(a \land b) = h(a) \land h(b) \]

Boolean relations

\[ R \subseteq X_1 \times X_2 \]
\[ R(x) \text{ closed subset of } X_2, \]
\[ h_R(U) = \{ x : R(x) \subseteq U \} \in \text{Clop}(X_1) \]
\[ (x, y) \in R_h \text{ sii } h^{-1}(x) \subseteq y \]

A **round filter** of S4-subordination algebra \( \langle A, \prec \rangle \) is a filter \( F \) satisfying the following condition:

\[ a \in F \implies \exists b \in F \ (b \prec a). \]
Topological characterizations

Let $h : A_1 \to A_2$ be a meet-homomorphism between two $S4$-subordination algebras. Let $\langle X_1, \tau_1, \tau_{S_1} \rangle$ and $\langle X_2, \tau_2, \tau_{S_2} \rangle$ be the bitopological subordination spaces. Let $R = R_h$ be the Boolean relation defined by $h$.

1. The following conditions are equivalent:
   (a) $a \prec b$ implies $h(a) \prec h(b)$.
   (b) $h_R(\text{int}_{S_1}(U)) \subseteq \text{int}_{S_2}(h_R(U))$, for each $U \in \text{Clop}(X_1)$.

2. The following conditions are equivalent:
   (a) $a \prec h(b) \Rightarrow \exists c \ (a \leq h(c) \text{ and } c \prec b)$.
   (b) $h^{-1}(F)$ is a round filter of $A_1$, for each round filter $F$ of $A_2$,
   (c) $\text{int}_{S_2}(h_R(U)) \subseteq h_R(\text{int}_{S_1}(U))$, for all $U \in \text{Clop}(X_1)$.

3. The following conditions are equivalent:
   (a) $a \prec b \Rightarrow \neg h(\neg a) \prec h(b)$, for all $a, b \in A_1$.
   (b) $g_R(\text{int}_{S_1}(U)) \subseteq \text{int}_{S_2}(h_R(U))$, for all $U \in \text{Clop}(X_1)$,
where $g_R(U) = h_R(U^c)^c$. 
We recall that a map \( h : A \to B \) is a Boolean homomorphism iff

1. \( h^{-1}(F) \) of \( A_1 \) is a filter for each filter \( F \) of \( F_2 \).
2. \( h^{-1}(P) \) is a maximal filter of \( A_1 \) for each maximal filter of \( A_2 \).

In the context of \( S5 \)-subordination algebras we have a similar result.

\[
\text{SubS5} = \text{SubS4} + \{ a \prec b \Rightarrow \neg b \prec \neg a \}.
\]

**Theorem**

Let \( h : A_1 \to A_2 \) be a meet-homomorphism between two \( S5 \)-subordination algebras. Then the following conditions are equivalent:

1. \( h \) satisfies the following conditions:
   1. \( a \prec b \Rightarrow \neg h(\neg a) \prec h(b) \), for all \( a, b \in A_1 \).
   2. \( a \prec h(b) \Rightarrow \exists c \ (a \leq h(c) \text{ and } c \prec b) \).

2. \( h \) satisfies the following conditions:
   1. \( h^{-1}(F) \) is a round filter of \( A_1 \) for each round filter \( F \) of \( A_2 \).
   2. \( h^{-1}(P) \) is a maximal round filter of \( A_1 \) for each maximal round filter \( P \) of \( A_2 \).
Categorical dualities

Thus, we can define different categories taking the same objects but with different morphisms.

SubS4

- **Objects:** $S4$-subordination algebras
- **Morphisms:** Meet-homomorphisms $h : A_1 \rightarrow A_2$ satisfying the conditions:
  - $a \ll b$ implies $h(a) \ll h(b)$,
  - $a \ll h(b) \Rightarrow \exists c (a \leq h(c) \text{ and } c \ll b)$.

BitoSub

- **Objects:** bitopological subordination spaces
- **Morphisms:** Boolean relations $R \subseteq X_1 \times X_2$ satisfying the condition:
  - $\text{int}_{S_1}(h_R(U)) = h_R(\text{int}_{S_2}(U))$, for every $U \in \text{Clop}(X_2)$.

Theorem

The categories **SubS4** and **BitoSub** are dually equivalent.

The category **SubS4** is related to the category of quasi-topological algebras introduced in Celani S. A.: Quasi-Modal algebras, Mathematica Bohemica Vol. 126, No. 4 (2001),
Bitopological duality for de Vries algebras

DeV

- **Objects:** de Vries algebras
- **Morphisms:** de Vries homomorphisms, i.e., meet-homomorphisms $h : A_1 \rightarrow A_2$ satisfying the conditions:
  - $a \prec b$ implies $\neg h(\neg a) \prec h(b)$,
  - $h(a) = \bigvee \{h(b) : b \prec a\}$.

BitoDeV

- **Objects:** Bitopological de Vries spaces, i.e., bitopological subordination spaces $\langle X, \tau, \tau_S \rangle$ satisfying the conditions:
  - $\langle X, \tau \rangle$ is extremally disconnected.
  - $\triangleleft_S$ is an equivalence relation.
  - $\text{int}_S(U) \neq \emptyset$, para cada $U \in \text{Clop}(X) - \{\emptyset\}$
- **Morphisms:** de Vries relations, i.e., Boolean relations $R \subseteq X_1 \times X_2$ satisfying the condition:
  - $R(x) \neq \emptyset$, for each $x \in X_1$,
  - $g_R(\text{int}_S_2(U)) \subseteq \text{int}_S_1(h_R(U))$ for every $U \in \text{Clop}(X_2)$.
  - $h_R(U) = \text{cl}(h_R(\text{int}_S_1(U)))$, for every $U \in \text{Clop}(X_2)$.
The categories $\text{Dev}$ and $\text{BitoDeV}$ are dually equivalent.

We note that the category $\text{DeV}$ is the category of complete compingent algebras study by de Vries in H.: Compact spaces and compactifications. An algebraic approach, PhD thesis, University of Amsterdam, 1962.
Remarks

1. The usual composition of de Vries homomorphism is not a de Vries homomorphism, and
2. the usual composition of de Vries relations is not a de Vries relation.

Thus, we need to define new compositions.

- The definition of the proper composition between de Vries homomorphism is given by de Vries


- The definition of the proper composition between de Vries relations is given in

An application. Subordination congruences

Definition

Let \( \langle A, \prec \rangle \) be a subordination algebra. A Boolean congruence \( \theta \subseteq A \times A \) is a subordination congruence, or quasi-modal congruence, if \( \theta \) satisfies the following condition.

If \( c \prec a \) and \( (a, b) \in \theta \), then there exists \( d \in A \) such that \( d \prec b \) and \( (c, d) \in \theta \).

\[
\prec \circ \theta \subseteq \theta \circ \prec.
\]

We can prove that

- The lattice of subordination congruences is isomorphic to the lattice of round filters,
- Any subordination congruence is the kernel of a Boolean homomorphism \( h : A \rightarrow B \) satisfying the conditions:
  1. \( a \prec b \) implies \( h(a) \prec h(b) \),
  2. \( a \prec h(b) \Rightarrow \exists c \ (a \leq h(c) \text{ and } c \prec b) \).
We recall that any closed subset $Y$ of a Stone space $\langle X, \tau \rangle$ define a Boolean congruence

$$\theta(Y) = \{(a, b) \in A \times A : \varphi(a) \cap Y = \varphi(b) \cap Y\}.$$ 

A subset $Y \subseteq X$ is saturated of $\langle X, \tau_S \rangle$ if

$$Y = \bigcap\{O \in \tau_S : Y \subseteq O\}.$$
Theorem

The lattice of subordination congruences of a S4-subordination algebra $\langle A, \prec \rangle$ is isomorphic to the lattice of subsets which are closed in the Stone topology $\tau$ and saturated in the topology $\tau_S$:

$$\text{Con} \langle A, \prec \rangle \cong C \langle X, \tau \rangle \cap S \langle X, \tau_S \rangle.$$
Conclusion

In this talk we have shown that there exists bitopological dualities for some classes of Boolean algebras with a subordination. In particular, we have shown a new duality for de Vries algebras.

1. **DeV**: Category of de Vries algebras

\[ \text{DeV} \leftrightarrow \text{BitDeV} \leftrightarrow \text{KHAUS} \leftrightarrow \text{KRFrm} \leftrightarrow \text{Gle} \]

- **BitDeV**: Bitopological de Vries spaces;
- **KHAUS**: Category of compact and Hausdorff spaces (de Vries duality);
- **KRFrm**: Category of compact regular frames (Isbell theorem);
- **Gle**: Category of Gleason spaces (Stone spaces with an equivalence relation satisfying certain conditions).
For future work

(1) We want to give a direct proof of the dual equivalence

\[ \text{BitoDeV} \cong^d \text{KRFrm}. \]

(2) The definition of subordination relation does not use Boolean negation. So, we can study bitopological dualities to bounded distributive lattices and Heyting algebras with a S4-subordination relation.

(3) A common generalization of de Vries algebras and stably compact frames is the class \( \text{PrFrm} \) of \textit{proximity frames}. This class is studied in Guram Bezhanishvili and John Harding, \textit{Proximity Frames and Regularization}, Appl Categor Struct (2014) 22:43–78.

We suggest developing a bitopological duality for the class of proximity frames.
Thanks

Muchas Gracias

Please, for more details write to sergiocelani@gmail.com. I will be very happy to answer