On Finitely-Generated Johansson Algebras

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Introduction

In 1973, A.V. Kuznetsov - without a proof - announced a theorem stating that given a finitely-generated Heyting algebra A, a sublattice of all dense elements of A contains the smallest element and can be regarded as a Heyting algebra, which is also finitely-generated. This theorem entails that any infinite finitely-generated Heyting algebra contains an infinite linearly ordered (alias chain) subalgebra and hence, if a variety of Heyting algebras does not contain an infinite chain algebra, this variety is locally finite.

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A proof was given in 1986 by A. Citkin and also, in 2005 by G. Bezhanishvili and R. Grigolia.

We generalize this theorem and extend it to the Johansson algebras.

Introduction

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Kolmogorov suggested to replace axioms for negation with a single axiom $(p \to q) \to ((p \to \neg q) \to \neg p)$.

Johansson suggested to drop axioms for negation and instead, use a new primitive nulary connective $\curlywedge$, without any axioms for it, and let $\neg p := p \to \curlywedge$. Let us note that Kolmogorov's axiom is derivable in Johansson's calculus..

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Variety of *Brouwerian algebras* (or implicative lattices) can be viewed as as subvariety of J-algebras defined by identity

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J-Algebras

If A is a J-algebra and $a \in A$, by $[a)$ we denote a *principal filter* of A generated by a, that is $[a) = \{b \in A \mid a \leq b\}$.

Similarly to Heyting algebras, for any coset $C \in A/[a)$,

(a) C contains the largest element denoted by m_C;

(b) $card(C) \leq card([a))$.

J-Algebras

By a *rank* of element $a \in A$, we understand the cardinality of $A/[a)$, that is, the rank of a is the cardinality of the ideal $\{b \in A \mid b \leq a\}$.

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In addition, we can convert $[a)$ into J-algebra (denoted by $A[a]$) by letting $f = a \vee f_A$.

Thus, if $a \in A$ is an element of finite rank, then, A is finite as long as $A[a]$ is finite.

J-Algebras

Theorem

Let A *be a J-algebra generated by elements* $G \subseteq A$*, and* $a \in A$*. Then J-algebra* $A[a]$ *is generated by elements*

$$G' \rightleftharpoons \{a, a \vee g, a \vee m_C \mid g \in G, C \in A/[a)\}.$$

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Immediately from Theorem 1, we obtain the Generalized Kuznetsov Theorem.

Theorem

If A *is a finitely-generated J-algebra and* $a \in A$ *is an element of a finite rank, then J-algebra* $A[a]$ *is finitely-generated.*

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Every finitely-generated J-algebra has the smallest element denoted by m_A, and we let $\neg a := a \to m_A$. Thus, $(A; \to, \wedge, \vee, 1, m_A)$ is a Heyting algebra.

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Accordingly, an element a of a finitely-generated J-algebra A is *dense* if $\neg a = m_A$. All dense elements of a finitely-generate J-algebra A form a filter denoted by D(A).

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Corollary (Kuznetsov's Theorem)

If A *is finitely-generated J-algebra, then filter* $D(A)$ *contains the smallest element* d_A *and algebra* $A[d_A]$ *is finitely-generated.*

J-Algebras

Corollary

Let a *be an atom of a finitely-generated J-algebra* A. *Then*

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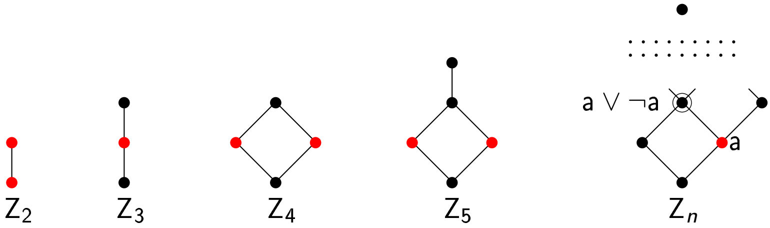
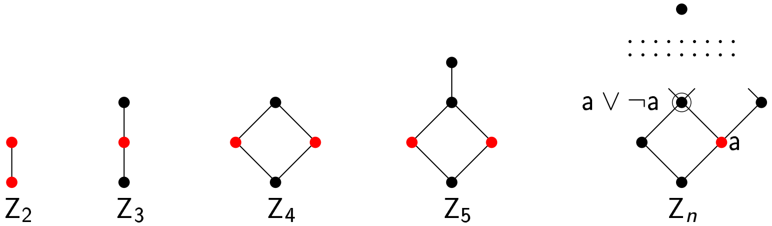
In every single-generated Heyting algebra, every distinct from 0 generator is an atom. Thus, if A is a Heyting algebra generated by element a, then A[a] is generated by a single element $a \vee \neg a$.



J-Algebras

Using Corollary 4, by simple induction (and without use of the Nishimura Theorem) one can prove that for every $n > 1$ there is a unique modulo isomorphism single-generated Heyting algebra of cardinality n.



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Every nontrivial finitely-generated J-algebra contains the smallest dense element $\Delta 0 = (f \vee \neg f) \wedge \bigwedge_{g \in G}(g \vee \neg g)$, *where* G *is a finite set of generators of* A.

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Proposition

Suppose that A *is an n-generated J-algebra. Then,*

$$r_A(\Delta 0) \leq 2^{2^{n+1}}.$$

J-Algebras

Let A be a finitely generated J-algebra. Define by induction

$$\Delta^0 0 \rightleftharpoons 0, \quad \Delta^1 0 \rightleftharpoons \Delta 0, \quad \Delta^{n+1} 0 \rightleftharpoons \Delta\Delta^n 0.$$

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Corollary

A finitely-generated J-algebra A is finite if and only if there is a natural number n such that $\Delta^n 0 = 1$.

J-Algebras

Suppose that A is a J-algebra and $a, b \in A$. Element a is *strongly smaller* than element b (in symbols $a \ll b$) if

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Suppose that A is a *J*-algebra and $a, b \in A$. Element a is *strongly smaller* than element b (in symbols $a \ll b$) if

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Relation $\ll$ is transitive and asymmetric with a single exception: $1 \ll 1$.

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Refined height of element a (in symbols, $\mathfrak{h}(\mathsf{a})$) is a maximal length of strongly descending chains of elements $\mathsf{a}_1 \ll \cdots \ll \mathsf{a}_n \leq \mathsf{a}$ smaller than a less 1.

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Refined height of algebra is the refined height of 1. Note that $\mathfrak{h}(\mathsf{a}) > 0$ if and only if a is dense.

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Refined height of algebra is the refined height of 1. Note that $\mathfrak{h}(a) > 0$ if and only if a is dense.

Refine height of trivial algebra (and only it) is 0. Refine height of Boolean algebras is 1.
For Heyting algebras of finite slices, refine height is the minimal number of slice this algebra belongs to.

J-Algebras

Proposition

Let A *be a J-algebra, and* $\mathsf{F} \subseteq \mathsf{A}$ *be a filter. Then, for any* $\mathsf{a}, \mathsf{b} \in \mathsf{A}$ *such that* $\mathsf{a} \notin \mathsf{F}$*,*

(a) *if* $\mathsf{a} \ll \mathsf{b}$*, then* $\mathsf{a}/\mathsf{F} \ll \mathsf{b}/\mathsf{F}$*;*

(b) *if* $\mathsf{a}/\mathsf{F} \ll \mathsf{b}/\mathsf{F}$*, then there are* $\mathsf{a}' \in \mathsf{a}/\mathsf{F}$ *and* $\mathsf{b}' \in \mathsf{b}/\mathsf{F}$ *such that* $\mathsf{a}' \ll \mathsf{b}'$*.*

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(a) *if* $a \ll b$, *then* $a/F \ll b/F$;

(b) *if* $a/F \ll b/F$, *then there are* $a' \in a/F$ *and* $b' \in b/F$ *such that* $a' \ll b'$.

There is a big difference between height and refined height:

$$h(a) \geq h(a/F) \text{ while } \mathfrak{h}(a) \leq \mathfrak{h}(a/F).$$

It is possible that $\mathfrak{h}(a)$ is finite, while $\mathfrak{h}(a/F)$ is infinite.

J-Algebras

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Theorem

Suppose a J-algebra A *is finitely-generated. Then,* A *is finitely approximable if and only if every distinct from* 1 *element of* A *has a finite refined height.*

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Suppose a J-algebra A is finitely-generated. Then, A is finitely approximable if and only if every distinct from 1 element of A has a finite refined height.

The condition that J-algebra A is finitely-generated is essential: free Heyting algebra $F_H(\omega)$ is finitely approximable, but contains some elements of infinite refined height.

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The condition that J-algebra A is finitely-generated is essential: free Heyting algebra $F_H(\omega)$ is finitely approximable, but contains some elements of infinite refined height.
On the other hand, in $F_H(n)$, all distinct from 1 elements have finite refined height.

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A finitely-generated J-algebra A *is finite if and only if the height of* A *is finite.*

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A finitely-generated J-algebra A *is finite if and only if the height of* A *is finite.*

Corollary

In a finitely-generated J-algebra A*, element* a *has a finite rank if and only if it has a finite height.*

J-Algebras

Final remark.

For any element a of a finitely-generated J-algebra A having a finite height, algebra A[a] is finitely generated and contains the smallest dense element which we denote Δa. It is not hard to see that Δa is the same element as the one defined in KM-algebras. Therefore, the preceding statement is equivalent to a statement formulated by Kuznetsov in 1982, the proof of which, to the best of my knowledge, has never been published.

Thanks

Thank you for your attention.

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