The Equational Theory of Distributive Lattice-Ordered Monoids

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ORDER-ENDOMORPHISM MONOIDS

EXAMPLE

Let Ω denote a totally ordered set (=chain).

Let $\mathrm{End}(\Omega)$ be the set of order-endomorphisms of Ω, with monoid operation given by function composition.

The order on Ω 'lifts' to a partial order on $\mathrm{End}(\Omega)$, as follows:

$$f \leq g \quad \Longleftrightarrow \quad f(\omega) \leq g(\omega) \text{ for all } \omega \in \Omega.$$

The following result holds in this context.

THEOREM (ANDERSON AND EDWARDS, 1984)

Every distributive lattice-ordered monoid is isomorphic to a substructure of $\mathrm{End}(\Omega)$*, for some chain* Ω*.*

DISTRIBUTIVE LATTICE-ORDERED MONOIDS

A distributive lattice-ordered monoid (shortly, distributive ℓ-monoid) is a monoid and a *distributive* lattice such that the monoid multiplication distributes over the lattice operations.

EXAMPLE

- The additive monoids $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$, equipped with lattice operations $\min$ and $\max$ are commutative distributive ℓ-monoids.
- Every ℓ-group (with operations $\wedge, \vee, \cdot, \mathrm{e}$) is a distributive ℓ-monoid.

The class of distributive ℓ-monoids is a *variety* in the language $\wedge, \vee, \cdot, \mathrm{e}$.

The variety is generated by the class $\{\mathrm{End}(\Omega) \mid \Omega \text{ is a chain}\}$.

NOTE: it suffices to consider equations of the form:

$$s_1 \wedge \cdots \wedge s_m \leq t_1 \vee \cdots \vee t_n,$$

where $s_1, \ldots, s_m, t_1, \ldots, t_n$ are monoid terms.

The Finite Model Property

Theorem (A. Colacito, N. Galatos, G. Metcalfe, S. Santschi)
The variety of all distributive ℓ-monoids is generated by the class

$$\{\mathrm{End}(\Phi) \mid \Phi \textit{ is a finite chain}\}.$$

Corollary (A. Colacito, N. Galatos, G. Metcalfe, S. Santschi)
The equational theory of distributive ℓ-monoids is decidable.

Main Question

What is the relationship between the equational theory of distributive ℓ-monoids and the equational theory of ℓ-groups?

Conservative Extension

The corresponding question is known to have a negative answer for abelian ℓ-groups and commutative distributive ℓ-monoids. (Repnitskiĭ, 1983)

Also:

Theorem (A. Colacito, N. Galatos, G. Metcalfe, S. Santschi)

There is an 'inverse-free' equation that is valid in all totally ordered groups, but not in all totally ordered monoids.

Conservative Extension

Theorem (A. Colacito, N. Galatos, G. Metcalfe, S. Santschi)

The variety of all distributive ℓ-monoids is generated by the ℓ-monoid $\mathrm{Aut}(\mathbb{Q})$.

- The equational theory of ℓ-groups is a conservative extension of the equational theory of distributive ℓ-monoids.
- The variety of distributive ℓ-monoids is the variety generated by the 'inverse-free' reducts of ℓ-groups.
- Distributive ℓ-monoids and ℓ-groups satisfy exactly the same equations in the language of ℓ-monoids.

EXAMPLE

Let $\mathrm{End}(2)$ be the distributive ℓ-monoid of endomorphisms of $\langle \{0, 1\}, \leq \rangle$.

Let $\langle k_0, k_1 \rangle$ denote the member of $\mathrm{End}(2)$ with $0 \mapsto k_0$ and $1 \mapsto k_1$.

The equation $xy \leq yx$ fails in $\mathrm{End}(2)$, as the latter is not commutative.

Example

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Let $\langle k_0, k_1\rangle$ denote the member of $\mathrm{End}(2)$ with $0 \mapsto k_0$ and $1 \mapsto k_1$.

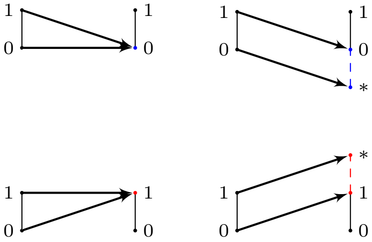
The equation $xy \leq yx$ fails in $\mathrm{End}(2)$, as the latter is not commutative. Indeed,

$$(\langle 1,1\rangle \circ \langle 0,0\rangle)(0) = 1 \not\leq 0 = (\langle 0,0\rangle \circ \langle 1,1\rangle)(0)$$

EXAMPLE

We want to find automorphisms of $\mathbb{Q}$ that falsify the equation $xy \leq yx$.

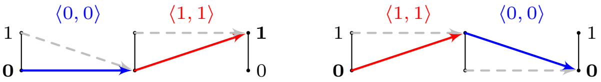
The main idea is the following:



How do they interact, e.g., in the composition $\langle 1, 1 \rangle \circ \langle 0, 0 \rangle$ and $\langle 0, 0 \rangle \circ \langle 1, 1 \rangle$?

Example

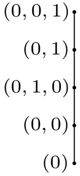
It helps to think of each term as a set of 'paths';



namely:

$$(0), (0, 0), (0, 0, 1), (0), (0, 1), (0, 1, 0),$$

and order them lexicographically (from the right):



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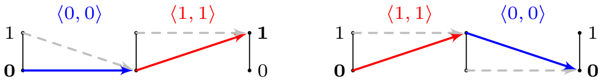

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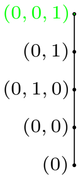

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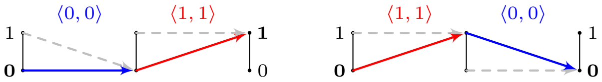

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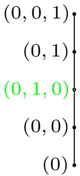

This makes sure that the ‘path’ $(\langle 1,1\rangle \circ \langle 0,0\rangle)(0)$ is bigger than the ‘path’ $(\langle 0,0\rangle \circ \langle 1,1\rangle)(0)$.

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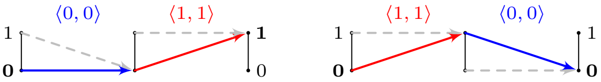

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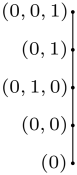

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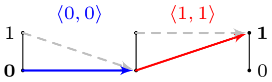
Example

We consider partial maps on Φ defined as follows:

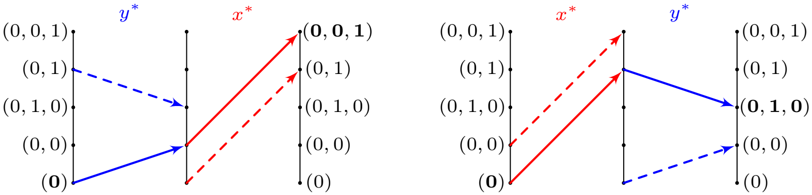
$$x^* = \{((w_0, \dots, w_k), (w_0, \dots, w_k, w_{k+1})) \in \Phi \times \Phi \mid w_{k+1} = \langle 1, 1 \rangle (w_k)\}.$$

$$y^* = \{((w_0, \dots, w_k), (w_0, \dots, w_k, w_{k+1})) \in \Phi \times \Phi \mid w_{k+1} = \langle 0, 0 \rangle (w_k)\}.$$

The resulting representation of the terms goes from:



to:



Forgetting the Inverse

To check whether an equation is valid in all distributive ℓ-monoids, it suffices to check the validity of this same equation in all ℓ-groups.

A certain converse holds.

It suffices to consider $\mathrm{e} \leq t_1 \vee \cdots \vee t_n$, where $t_1, \ldots, t_n$ are group terms.

For example, take $\mathrm{e} \leq t_1 \vee t_2$ with variables in X.
We have the following equivalences with respect to ℓ-groups.

1) $\mathrm{e} \leq t_1 \vee tx^{-1} \iff x \leq t_1 x \vee t$

2) $\mathrm{e} \leq t_1 \vee tx^{-1}s \iff \mathrm{e} \leq t_1 \vee ty \vee y^{-1}x^{-1}s$ for $y \notin X$

$\iff xy \leq xyt_1 \vee xyty \vee s$ for $y \notin X$

Corollary (Holland and McCleary, 1979)

The equational theory of ℓ-groups is decidable.

RIGHT ORDERS

A right-invariant order $\leq$ on a monoid M is a total order on M s.t. for all $a, b, t \in M$,

$$a \leq b \implies at \leq bt.$$

We call a *right-invariant* order on M a right order on M.

A monoid that admits a right order is said to be *right-orderable*.

EXAMPLE

- Finite groups are not right-orderable.
- Abelian groups are (right-)orderable if and only if they are torsion-free. (Levi, 1942)
- Free groups are right-orderable. (Birkhoff, Iwasawa, Neumann, Shimbireva, Vinogradov, Bergman...)
- A countable group is right-orderable if and only if it acts faithfully on $\mathbb{R}$ by orientation-preserving homeomorphisms.

RIGHT ORDERS

THEOREM (COHN 1957; CONRAD 1959)

A group is right-orderable if and only if is a subgroup of an ℓ-group.

The same does not hold for monoids and distributive ℓ-monoids.

The monoid $\mathrm{End}(\Omega)$ is not right-orderable for any chain Ω with $|\Omega| \geq 3$.

THEOREM (A. COLACITO, N. GALATOS, G. METCALFE, S. SANTSCHI)

Every right order on the free monoid over X extends to a right order on the free group over the same X.

CONCLUSIONS

- Finite Model Property ⇒ Decidability distributive lattice-ordered monoids.
- Generation by $\mathrm{Aut}(\mathbb{Q})$ ⇒ Lattice-ordered groups are a conservative extension.
- Effective translation ⇒ Decidability of ℓ-groups.
- Failure of conservative extension result for totally ordered monoids and groups.
- Extension of right orders on the free monoid over X to the free group over X.
- Axiomatization of the variety generated by totally ordered monoids.

A. Colacito, N. Galatos, G. Metcalfe, S. Santschi. From distributive ℓ-monoids to ℓ-groups, and back again, arXiv:2103.00146 [math.GR] 2021. (Submitted)

To do:

- Axiomatization of the ℓ-monoid fragment of totally ordered groups.
- Decidability of commutative distributive ℓ-monoids.
- Characterization of right-orderable monoids.
- Framework for a systematic study of the proof theory for ℓ-groups.