Countable Metric Spaces Without Isolated Points

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- Not direct!

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- Textbook proofs are rare!

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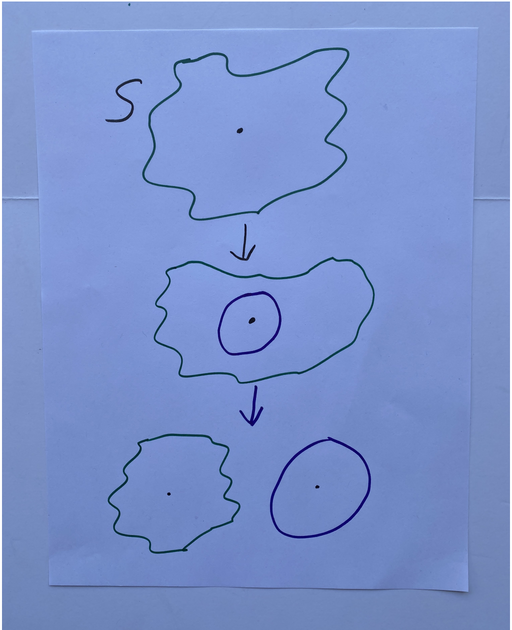
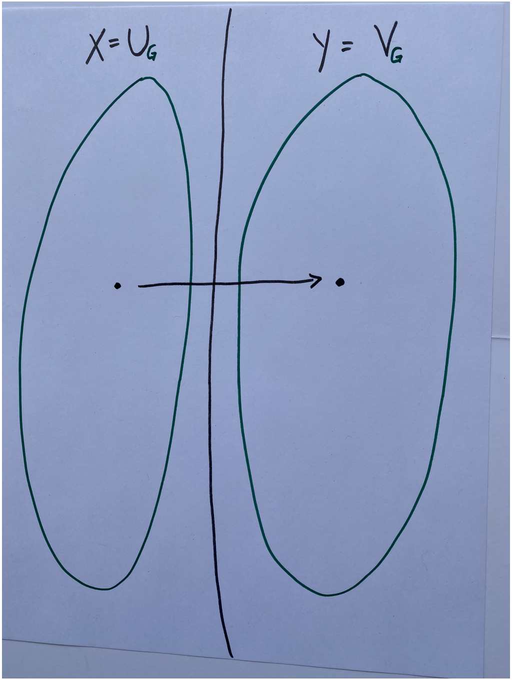
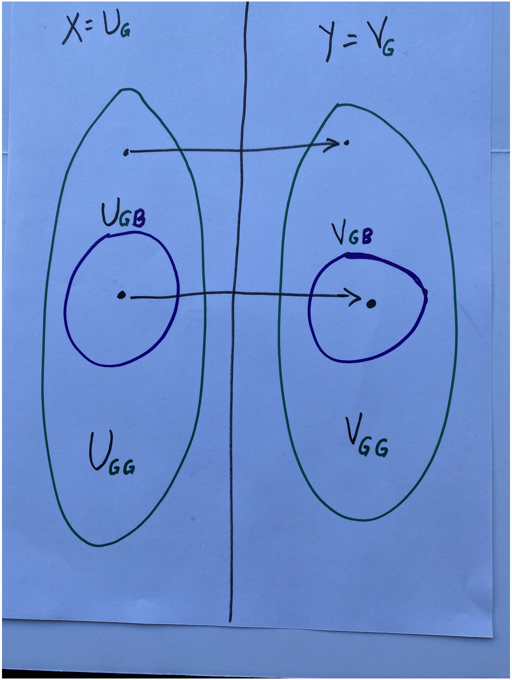
- $S \subset X, Y$: $\mathrm{next}(S) = x_k$ where $k = \min\{i : x_i \in S\}$

The Splitter




$X = U_G$
$Y = V_G$


X = U_G
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U_GB
V_GB
U_GG
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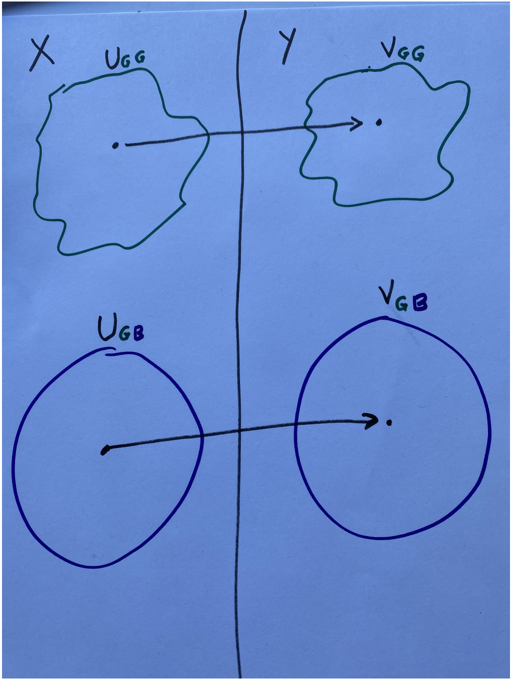
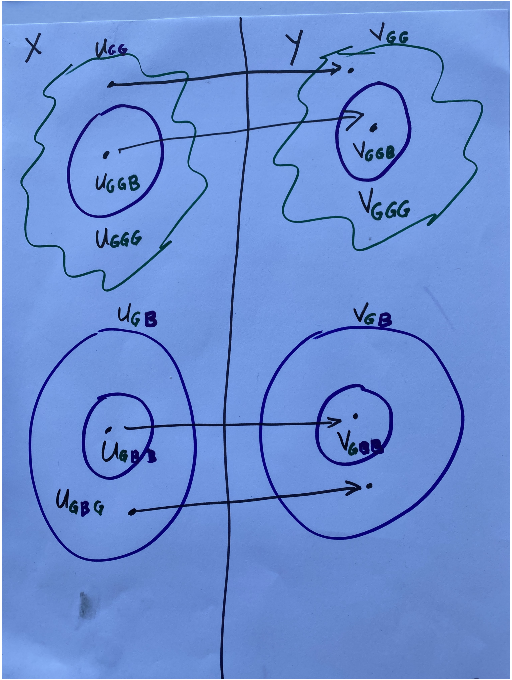


X
Y
UGG
VGG
UGB
VGB


X
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U_{GG}
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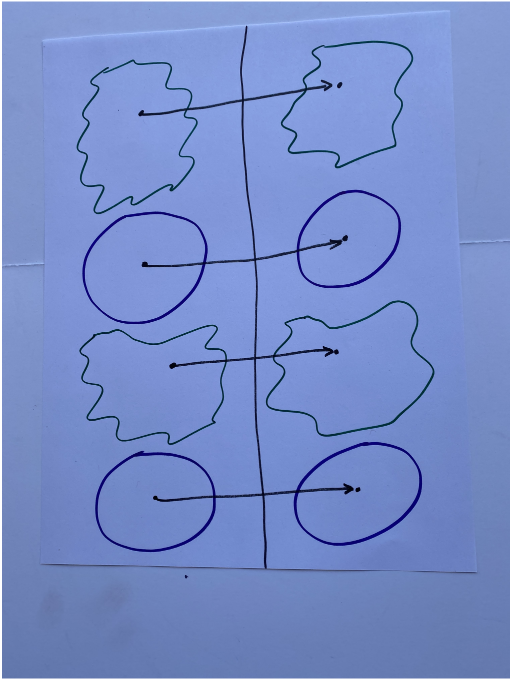

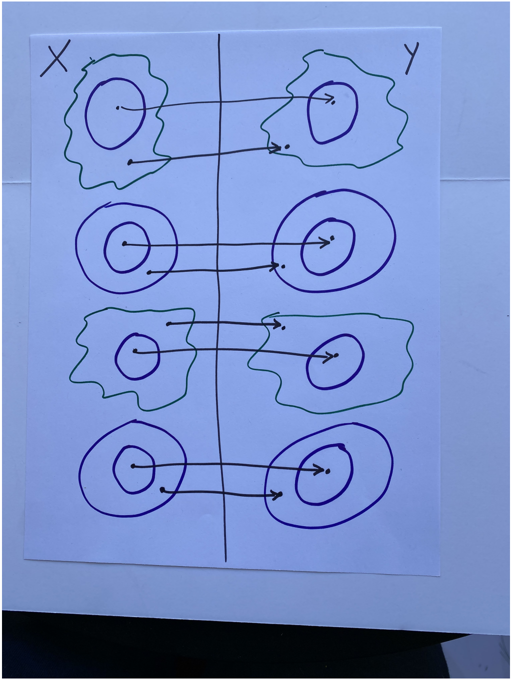
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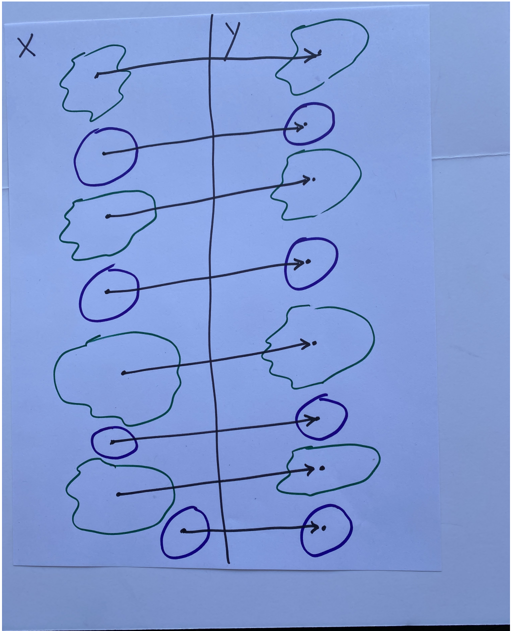
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References

Dasgupta, A. Countable metric spaces without isolated points. *Topology Explained*. June 2005, published by *Topology Atlas*. Accessed 7 June 2021 at http://dasgupab.faculty.udmercy.edu.

Dashiell, F., Countable metric spaces without isolated points. *Amer. Math. Monthly* 128(3) (2021), 265–267.