Interpolation and Beth definability in implicative fragments of IPC

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Based on a bachelor thesis supervised by N. Bezhanishvili and T. Moraschini

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Overview

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$$\phi \vdash \chi \vdash \psi.$$

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$\vdash$ has the **infinite Beth (definibility) property** if whenever Γ is a set of formulas that defines a set Z of variables implicitly in terms of X, then Γ defines Z explicitly in terms of X.

Three Beth properties

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finite Beth $\Leftarrow$ infinite Beth

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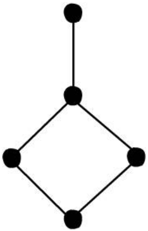

ISL's

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Amalgamation

An logic[1] has the **interpolation property** iff the corresponding variety has the **amalgamation property**:

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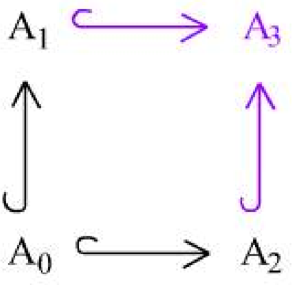

[1] algebraizable and over a proper class of variables

Epimorphism surjectivity

A logic[2] has the **infinite Beth property** iff the corresponding variety has **surjective epimorphisms** (**ES**).

[2] algebraizable and over a proper class of variables

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G. Bezhanishvili, Moraschini & Raftery 2017: $2^{\aleph_0}$, but not all, varieties of Heyting algebras with the **ES property**.

New results

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Similar classifications for varieties of ISL's and varieties of bounded ISL's.

How many?

Signature	$\wedge, \to$	$\wedge, \to, \bot$	$\wedge, \to, \bot, \vee$
Structure	**ISL**	**bounded ISL**	**Heyt. alg.**
Interpolation / **amalg.**	4	9	8 [3]
Proj. Beth / **strong ES**	8	30	16 [4]
Inf. Beth / **ES**	$2^{\aleph_0}$	$2^{\aleph_0}$	$2^{\aleph_0}$ [5]
Fin. Beth / **weak ES**	all[6]	all[6]	all[6]

[3] Maksimova 1977
[4] Maksimova 2000
[5] G. Bezhanishvili, Moraschini & Raftery 2017
[6] Kreisel 1960

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$$\alpha([p)) = [\alpha(p)).$$

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Why finite ISL's?

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Why finite ISL's?

ISL's are **locally finite** (Diego 1966), so Prf links every **variety of ISL's** to a unique class of **finite posets**.

Properties of Köhler duality

- **embedding**
- **homomorphic image**
- **product**

- **P-morphic image**
- **full poset embedding**
- **disjoint union**

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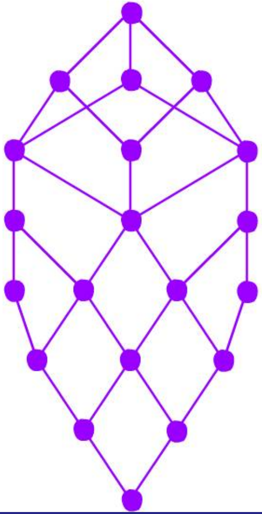
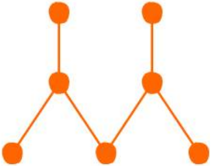
Again **local finiteness** is the reason for this.

Posets are smaller





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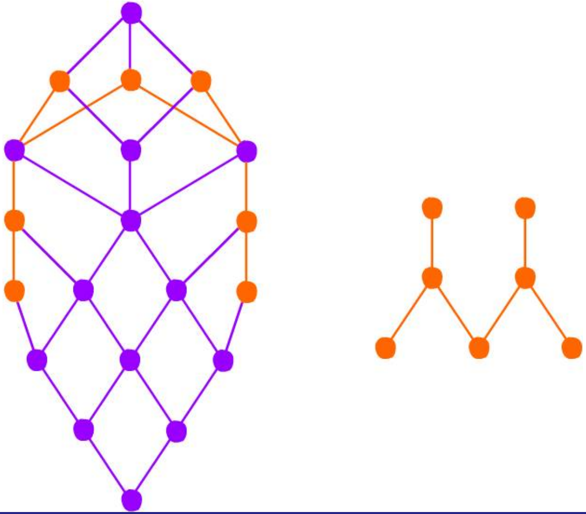

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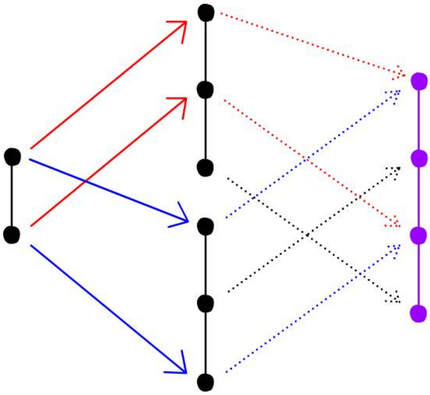
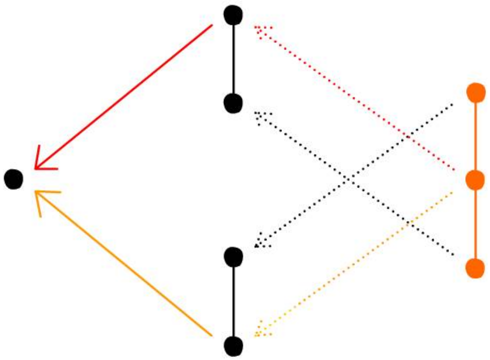
IPC $+bd_2$ has the interpolation property.
$\mathrm{IPC}_{\wedge,\to} + bd_2$ lacks the interpolation property.

Explanation with amalgamation

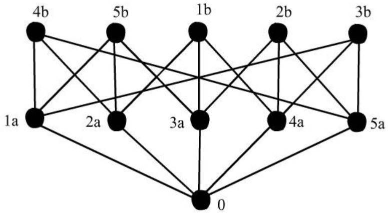


Explanation with **co-amalgamation**



$2^{\aleph_0}$ varieties with ES

Let P_5 be the **poset**:



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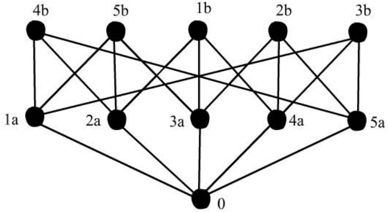

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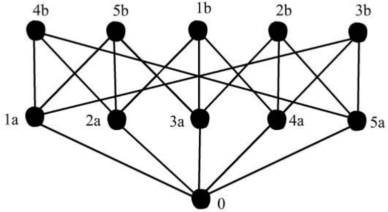

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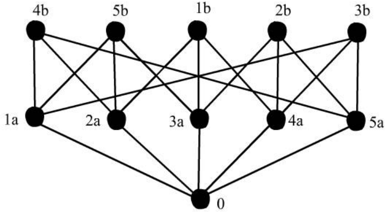

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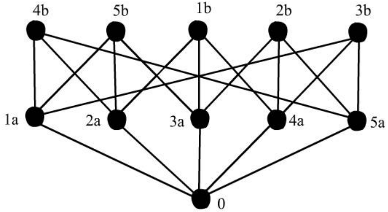

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These varieties are of **depth** 3 because P_n is of **depth** 3.

$2^{\aleph_0}$ varieties with ES

Theorem

Let $n < \omega$. Every variety consisting of (bounded) ISL's of depth $\leq n$, has the ES property.

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- a partial characterization of extensions of $IPC_{\wedge,\to(,\bot)}$ with the **infinite Beth property**.

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  - G. Bezhanishvili, Moraschini & Raftery: Epimorphisms in varieties of residuated structures (2017)
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- a generalization of the theorem “bounded depth $\Longrightarrow$ ES” to **arithmetical varieties**?
- number of varieties of **Heyting algebras** generated by the duals of **finite trees**?
- decidability of **local finiteness** of a finitely axiomatizable variety of **Heyting algebras**?

Thanks for listening.