Big Ramsey Degrees of Universal Limit Structures

Natasha Dobrinen*, University of Denver

joint work with

Kaiyun Wang†, Shaanxi Normal University

BLAST 2021

* Research supported by National Science Foundation.

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Fraïssé Classes

A class $\mathcal{K}$ of finite structures is a Fraïssé class if $\mathcal{K}$ has the

- hereditary property
- joint embedding property
- amalgamation property

We will be working with Fraïssé classes with finitely many binary relations and a linear order, such as

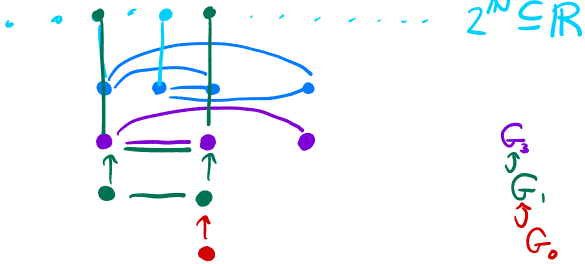
- finite ordered graphs.
- finite ordered directed graphs.
- finite ordered tournaments.
- finite ordered k-clique-free graphs.
- finite ordered graphs with n edge relations.

Universal Inverse Limit Structures

Let $\mathcal{K}$ be a Fraïssé class of ordered binary relational structures.

An inverse limit structure for $\mathcal{K}$ is a structure of size $2^{\aleph_0}$ which is obtained as the inverse limit $\varprojlim_{n\in\mathbb{N}} K_n$, where each K_n is a member of $\mathcal{K}$.

Example
$\mathcal{K}$ = all finite ordered graphs



A universal inverse limit structure for $\mathcal{K}$ is an inverse limit structure $\mathbf{K}$ into which every other inverse limit structure for $\mathcal{K}$ embeds continuously.

Remark. Universal inverse limit structures can be viewed as structures on perfect subsets of $\mathbb{R}$, with the inherited topology.

Ramsey Theory

Ramsey's Theorem. Given any $k, \ell \geq 1$ and a coloring of $[\mathbb{N}]^k$ into ℓ colors, there is an infinite subset $M \subseteq \mathbb{N}$ such that all members of $[M]^k$ have the same color.

Theorem (Blass). For each perfect subset P of $\mathbb{R}$ and each finite Baire measurable coloring of $[P]^n$, there is a perfect set $Q \subseteq P$ such that $[Q]^n$ has at most $(n-1)!$ colors.

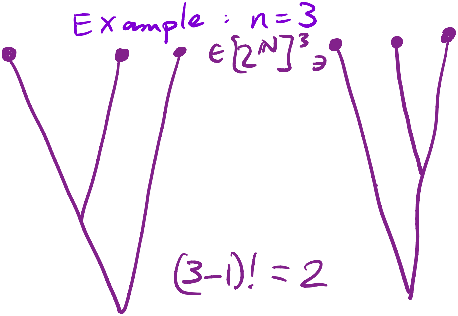
Theorem (Huber-Geschke-Kojman). For each finite ordered graph $\mathbf{A}$, there is a natural number $T(\mathbf{A}) \geq 1$ such that for each universal inverse limit graph $\mathbf{G}$, for each finite Baire measurable partition of $\binom{\mathbf{G}}{\mathbf{A}}$, there is a closed copy $\mathbf{G}' \subseteq \mathbf{G}$ of $\mathbf{G}$ such that $\binom{\mathbf{G}'}{\mathbf{A}}$ meets at most $T(\mathbf{A})$ parts.

Blass' Theorem and types

Theorem (Blass). For each perfect subset P of $\mathbb{R}$ and each finite Baire measurable coloring of $[P]^n$, there is a perfect set $Q \subseteq P$ such that $[Q]^n$ has at most $(n-1)!$ colors.

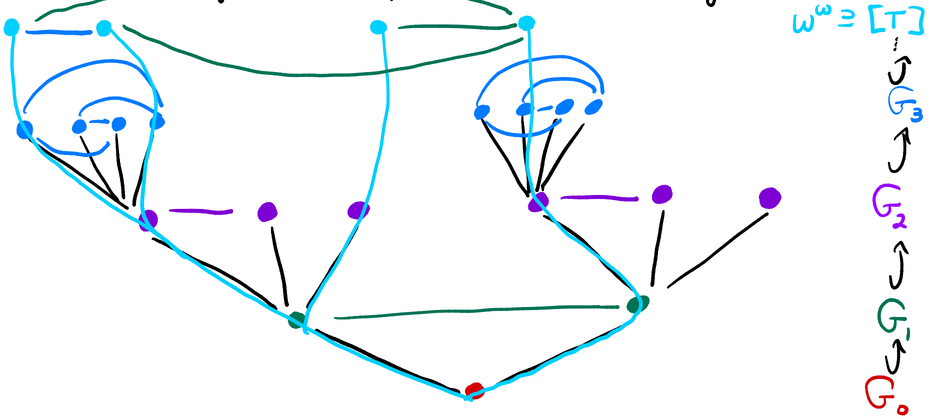
The $(n-1)!$ colors are given by types of finite subsets of $2^{\mathbb{N}}$.

Given $n \geq 1$, the types for n are the different n-size subsets of $2^{\mathbb{N}}$ s.t. all meets have different lengths



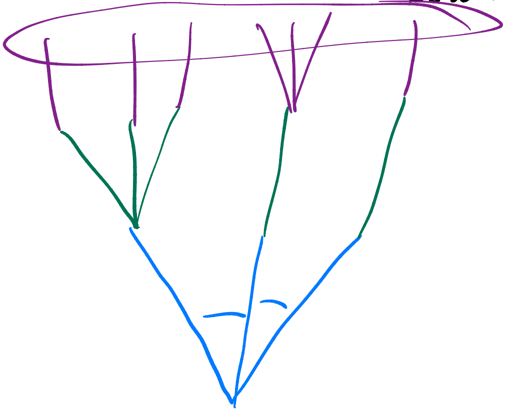
Constructions of universal inverse limit graphs

Construction of universal inverse limit graphs as limits of finitely branching trees: Take a sequence $G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \dots$ of finite ordered graphs s.t. $|G_n| = n+1$ and each f.o.g. embeds into some G_n.



Types in trees coding graphs

A type is a subtree s.t. each level has at most one splitting node. Note: The splitting degree is not bounded.



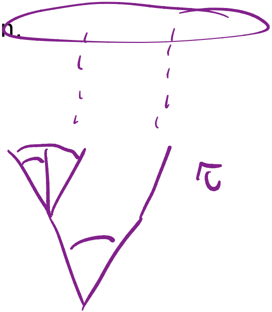
Make skew tree T s.t. [T] codes a universal limit graph.

Types and upper bounds for profinite graphs

Theorem (Zheng). For each type τ, there is a topological Ramsey space $\mathcal{R}(\tau)$ of trees T such that τ forms an initial segment of T and $[T]$ forms a universal profinite graph.

This recovers the theorem of Huber-Geschke-Kojman.

Given a finite graph G,
look at a given type τ
producing G



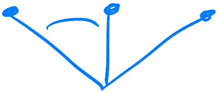
A space $(\mathcal{R}, \leq, r)$ is a topological Ramsey space if for any partition of $\mathcal{R}$ into two pieces with the Baire property, there is a $T \in \mathcal{R}$ such that $\{S \in \mathcal{R} : S \leq T\}$ is contained in one piece of the partition.

Exact big Ramsey degrees

Let $\mathbf{G}$ be a universal inverse limit graph, and let $\mathbf{A}$ be a finite ordered graph. We let $T(\mathbf{A})$ denote the smallest integer T such that for each finite Baire measurable partition of $\binom{\mathbf{G}}{\mathbf{A}}$, there is a closed copy $\mathbf{G}' \subseteq \mathbf{G}$ of $\mathbf{G}$ such that $\binom{\mathbf{G}'}{\mathbf{A}}$ meets at most T parts of the partition. We call $T(\mathbf{A})$ the big Ramsey degree of $\mathbf{A}$ in $\mathbf{G}$.

Theorem (D.-Wang). Given a universal profinite graph $\mathbf{G}$ and a finite ordered graph $\mathbf{A}$, the big Ramsey degree $T(\mathbf{A})$ equals the number of types representing a copy of $\mathbf{A}$.

We prove that each type persists in any smaller universal inverse limit.



Other universal limit structures

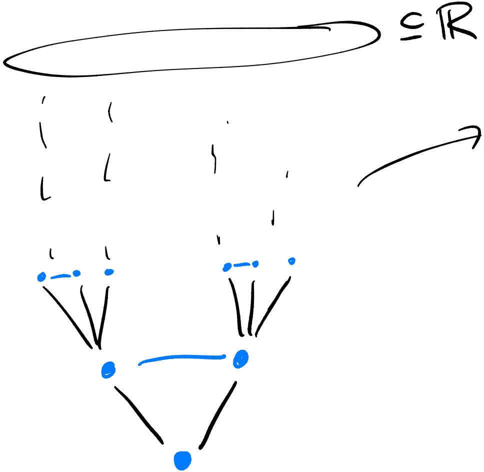
A Fraïssé class $\mathcal{K}$ has the Ramsey property if for each pair $\mathbf{A} \leq \mathbf{B}$ in $\mathcal{K}$ and any $r \geq 2$, there is a $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{B} \leq \mathbf{C}$ and for any r-coloring of $\binom{\mathbf{C}}{\mathbf{A}}$, there is a $\mathbf{B}' \leq \mathbf{C}$ with $\mathbf{B}' \cong \mathbf{B}$ such that all members of $\binom{\mathbf{B}'}{\mathbf{A}}$ have the same color.

Theorem (Nešetřil-Rödl). Many Fraïssé classes have the Ramsey property, including the classes of finite ordered graphs, directed graphs, k-clique-free graphs, and combinations of these.

Theorem (D.-Wang). Let $\mathcal{K}$ be a Fraïssé class of ordered structures with finitely many binary relations, and assume that $\mathcal{K}$ has the Ramsey property. Then for each type τ, there is a topological Ramsey space $\mathcal{R}(\tau)$ of trees T such that τ forms an initial segment of T and $[T]$ forms a universal inverse limit of $\mathcal{K}$.

Examples

triangle-free graph



take skew subtree: atmost one splitting node per level.

Exact big Ramsey degrees in universal limit structures

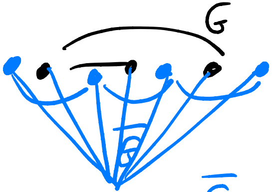
Theorem (D.-Wang). Let $\mathcal{K}$ be any of the classes of finite ordered graphs, finite ordered k-clique-free graphs, finite ordered directed graphs, or finite ordered tournaments. Then for each $\mathbf{A} \in \mathcal{K}$, the big Ramsey degree $T(\mathbf{A})$ for Baire measurable colorings of $\binom{\mathbf{K}}{\mathbf{A}}$ is exactly the number of types representing $\mathbf{A}$.

Similar methods seem to work for such structures with finitely many edges or directed edges. Details are being checked.

Examples

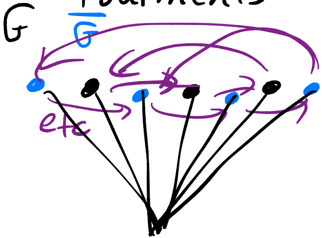
Lower Bound

graphs or k-cliquefree etc.



show that $\overline{G}$ persists in each sub-universal inverse limit structure.

tourments



Remarks: Halpern-Läuchli Theorem

Nešetřil-Rödl Theorem

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