Forcing, Stone spaces, and converging sequences

BLAST 2021

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A space is ω-free if it contains no converging sequences

A space is ω_1-free if it contains no converging ω_1-sequences (in the co-countable sense)

A $\beta\mathbb{N}$-free space contains no copy of $\beta\mathbb{N}$ (ω-free, not ω_1-free)

A space has countable tightness if $\overline{A} = \bigcup\{\overline{B} : B \in [A]^{\leq\aleph_0}\}$ and is sequential if $\overline{A} = A^{(\gamma)}$ for some γ, where, by recursion,

$$A^{(\alpha)} = \{x : (\exists\beta < \alpha)(\exists\langle a_n\rangle \to x) \ \ \{a_n\}_n \subset A^{(\beta)} \cup A\}$$

a space is Frechet if $A^{(1)} = \overline{A}$ (sequential order $=1$)

Part 1: a method of building free spaces (?and many others?)

Part 2: PFA methods for not having free spaces

investigations of these spaces

1. A Moore-Mrowka space is a compact space of countable tightness that is not sequential
2. An Efimov space is a compact space that is ω-free and $\beta\mathbb{N}$-free
3. Compact ω_1-free spaces that are not first-countable.
4. A compact sequential space that has sequential order > 2.

A compact space X is scattered if we can recursively "expose" the points one at at time:

x_α is exposed by compact open set $U_\alpha \subset \{x_\beta : \beta \leq \alpha\}$

x_α was exposed out of the "blob" $X \setminus \{x_\beta : \beta < \alpha\}$

Let's have this picture in mind

U_α is the set of purple points

just "prior" to exposure of x_α, U_α out of blob X^*_α

just notation, note no dependence on "blob"

B_α is the Boolean subalgebra of $\mathcal{P}(\{x_\beta : \beta < \alpha\})$ generated by $\{U_\beta : \beta < \alpha\}$ and $\mathcal{I}_\alpha$ is the ideal so generated

the dual filter $\mathcal{F}_\alpha = B_\alpha \setminus \mathcal{I}_\alpha$ is ultra.

X_α will be the Stone space $S(B_\alpha)$ and is homeomorphic to $X \ / \ X^*_\alpha$ (quotient space with blob collapsed)

and $\mathcal{F}_\alpha \in S(B_\alpha)$ corresponds to $[X^*_\alpha]$

Recursively create rather than uncover

The only thing special about U_α in the discovery process is that

$$U_\alpha \setminus \{x_\alpha\} \quad , \quad W_\alpha = \{x_\beta : \beta < \alpha\} \setminus U_\alpha$$

is a clopen partition of $\{x_\beta : \beta < \alpha\}$

i.e. any such $U_\alpha \setminus \{x_\alpha\}$ is a possibility including swapping with W_α

Our interests will require that $\{x_n : n \in \omega\} \notin \mathcal{I}_\alpha$

i.e. $\{x_n\}_n$ is dense in X_α for $\omega \leq \alpha$

When $\alpha < \omega_1$ and $\{\beta_n : n \in \omega\} \subset \alpha$ is any sequence such that $\langle x_{\beta_n} \rangle_n$ converges to $\mathcal{F}_\alpha$, U_α can be chosen so that $\langle x_n \rangle_n$ does not converge in $X_{\alpha+1}$. making suitable choices gives

Theorem (Juhasz-Weiss)

There is a thin-tall lcs $\{x_\alpha : \alpha < \omega_1\}$, i.e.

$$\{x_\alpha : \beta < \alpha < \omega_1\} \text{ is separable for all } \beta < \omega_1$$

Ostaszewski

Theorem

There is a suitable sequence $\{x_\alpha, U_\alpha : \alpha \in \omega_1\}$ so that

1. *$CH \vdash \mathcal{F}_{\omega_1}$ is not the limit of any converging sequence in X_{ω_1} (same as $X_{\omega_1} \setminus \{\mathcal{F}_{\omega_1}\}$ is sequentially compact, mini ω-free)*
2. *$\clubsuit \vdash X_{\omega_1}$ has countable tightness (and more)*

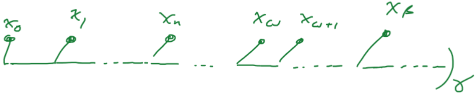
and therefore $\diamondsuit$ implies X_{ω_1} is a Moore-Mrowka space.

If CH fails and we have constructed X_α (some $\alpha \geq \omega_1$) and there is a sequence $\langle x_{\beta_n} \rangle_n \to \mathcal{F}_\alpha$, can we choose U_α to split that sequence (part of our exploration of ω-free)??

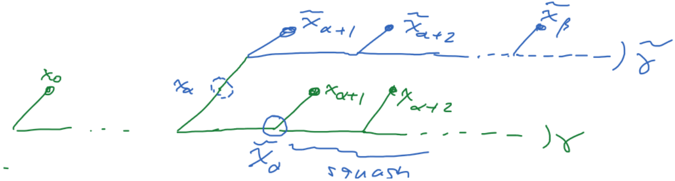
Not necessarily (e.g. $\mathcal{F}_\alpha$ could be top right corner of Tychonoff plank), but van Douwen showed that if $\mathfrak{b} = \mathfrak{c}$ we can build $X_\mathfrak{c}$

Minimal extensions and T-algebras: What if we have constructed

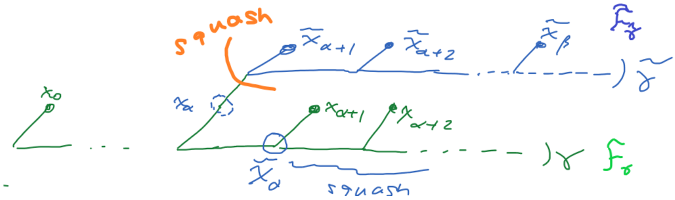
$\{x_\beta, U_\beta : \beta < \gamma\}$



and we jump back to some $\alpha < \gamma$ and decide to explore the W_α choice rather than the U_α one. In this new construction of $\{x_\beta : \beta < \alpha\} \cup \{\tilde{x}_\beta : \alpha < \tilde{\gamma}\}$, mentally collapse all of $\{\mathcal{F}_\gamma\} \cup \{x_\beta : \alpha < \beta < \gamma\}$ to the point $\tilde{x}_\alpha$ with $\tilde{U}_\alpha = W_\alpha \cup \{\tilde{x}_\alpha\}$ and continue to choose suitable $\tilde{U}_\beta$ $(\alpha < \beta)$ as we like.



Complete symmetry: the space X_γ can't see into the hole at x_α (filled with $\{\tilde{x}_\beta : \alpha < \beta < \tilde{\gamma}\} \cup \{\tilde{\mathcal{F}}_{\tilde{\gamma}}\}$) and the space $\tilde{X}_{\tilde{\gamma}}$ can't see in to the hole at $\tilde{x}_\alpha$ (containing $\{x_\beta : \alpha < \beta < \gamma\} \cup \{\mathcal{F}_\gamma\}$)



Also, neither knows (nor cares) which one came first.

They share the initial segment $\{x_\beta : \beta < \alpha\}$ and regard their space as the branching points off the main chain of growth

Use tree indexing rather than layers of tilde's

We can repeat the previous swap and plug (myopically collapse) (branching) process to our heart's content with a T-algebra

An admissible $T \subset 2^{<\theta}$ is a subtree that satisfies

$$(t^\frown 1)^\dagger = t^\frown 0 \in T \text{ iff } (t^\frown 0)^\dagger = t^\frown 1 \in T$$

Definition

A family $\{U_t : t \in T\}$ is a T-algebra if, for all $t^\frown 0 \in T$

1. $U_{t^\frown 1} = \{t^\frown 0\} \cup \left((t^\downarrow)^\dagger \setminus U_{t^\frown 0}\right)$, $\quad U_t = \emptyset$ for non-successor t
2. $U_{t^\frown 0} = \{t^\frown 1\} \cup \left((t^\downarrow)^\dagger \setminus U_{t^\frown 1}\right)$, $\quad$ (i.e. "$U_{t^\frown 0}, W_{t^\frown 0}$")
3. $\mathcal{S}_t = \{x_{(t\restriction \xi+1)^\dagger}, U_{t\restriction \xi+1} : \xi + 1 \in dom(t)\}$ is a suitable sequence. note that $(t \restriction \xi+1)^\dagger \in U_{t\restriction \xi+1} \in \mathcal{I}_{t\restriction \xi+1} \subset \mathcal{I}_t$,

use $B_t, \mathcal{I}_t, \mathcal{F}_t$ in place of $B_\alpha, \mathcal{I}_\alpha, \mathcal{F}_\alpha$

The space X_T

Let bT denote the set of all maximal branches of T. For each $z \in bT$, $\mathcal{S}_z = \{x_{(z\restriction\xi+1)^\dagger}, U_{z\restriction\xi+1} : \xi + 1 \in dom(t)\}$ is a suitable sequence and we loosely identify z with $\mathcal{F}_z$ in the space X_z.

The T-algebra topology is on $bT = X_T$

For $z \in bT$ and $\xi{+}1 \in dom(z)$, subbasic clopen in X_T:

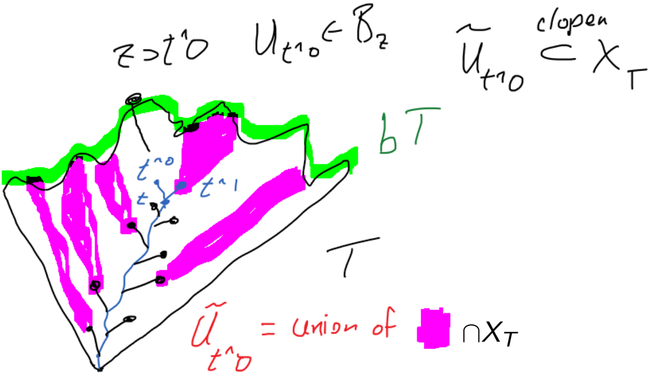
$$z \notin \widetilde{U}_{z\restriction\xi+1} = \{z' \in bT : (\exists \eta \leq \xi) \;\; z' \restriction \eta + 1 \in U_{z\restriction\xi+1}\}$$

we are just "exploding" all those collapsed swaps (i.e. $(t^\frown e)^\uparrow$)

$\widetilde{U}_{z\restriction\xi+1}$ only depends on $z\restriction\xi{+}1$ not on z

picture of $\widetilde{U}_{t^\frown 0}$

$$z \notin \widetilde{U}_{z\restriction\xi+1} = \{z' : (\exists \eta \le \xi)\ \ z' \restriction \eta + 1 \in U_{z\restriction\xi+1}\}$$



$\cap X_T$

comment: $t^\uparrow \cap bT$ $(= b\,t^\uparrow)$ is closed but rarely open

two examples

Cantor set

Let $T = 2^{<\omega}$ and $\{U_t : t \in T\}$ any choice. Then $X_T = bT = 2^\omega$ gets the canonical Cantor set topology, e.g. "$b\,t^\uparrow$" is clopen.

Double arrow on $\mathbb{C}$

Let $T = 2^{\leq\omega}$, $\{U_t : t \in 2^{<\omega}\}$ any choice, (yielding $\mathbb{C}$)
and $U_{z^\frown 0} = \{(z{\restriction}n{+}1)^\dagger : z(n) = 1\}$, then X_T is equal to the double arrow $\{z^\frown 0, z^\frown 1 : z \in 2^\omega\} \setminus \{\mathbf{0}^\frown 0, \mathbf{1}^\frown 1\}$,

i.e. $\{z'^\frown 0, z'^\frown 1\} \subset \widetilde{U}_{z^\frown 0}$ iff $z' <_{\text{lex}} z$ $\quad (z'(n) = 0, z(n) = 1)$

advantages of scattered but with greater scope

Facts about T-algebra spaces

1. A T-algebra space is compact separable (if we maintain, as promised, $\{x_{(t\restriction n+1)^\dagger} : n \in \omega\} \notin \mathcal{I}_t$) and has character $\leq ht(T)$
2. A T-algebra space is always $\beta\mathbb{N}$-free. Efimov here we come
3. A T-algebra space is ω-free iff for all $z \in bT$, $X_z \setminus \{\mathcal{F}_z\}$ is sequentially compact.

Major open Question

Is there an ω-free T-algebra space X_T? i.e. an Efimov space

ht$(T) > \mathfrak{p}$ is going to be necessary

"yes" would also solve Scarborough-Stone problem

Are there other ways to go beyond level ω?

Definition

if $\mathcal{A}$ is a T-algebra and t is a maximal branch, the poset $Part(\mathcal{A}, t)$ consists of disjoint pairs (a, b) from $B_t \cap \mathcal{I}_t$ ordered by coordinatewise extension.

Clearly $Part(\mathcal{A}, t)$ forces a generic choice for $\{U_{t\frown 0}, U_{t\frown 1}\}$ which will split all existing sequences converging to $\mathcal{F}_t$

If t is countable, $Part(\mathcal{A}, t)$ is countable. If t is uncountable, it may not be ccc.

Theorem (Koszmider)

If G is $Fn(\kappa, 2)$-generic and $\{\alpha_t : t \in 2^{<\omega_1} = T\} \subset \kappa$ is listed order-preserving such that $t \in V[G_{\alpha_t}]$ and $\{U_{t\frown 0}, U_{t\frown 1}\} \in \mathcal{A}$ is chosen from a $Part(\mathcal{A}_t, t)$ filter that is generic over $V[G_{\alpha_t}]$, then X_T is Efimov and Moore-Mrowka (basically Fedorchuk's space)

Lemma (Stone Duality)

If T_1 is a subtree of T_2 and $\mathcal{A}_{T_2} = \{U_t : t \in T_2\}$ is a T-algebra, then so is $\mathcal{A}_{T_1} = \mathcal{A}_{T_2} \restriction T_1 = \{U_t : t \in T_1\}$, and $f : X_{T_2} \to X_{T_1}$ defined by $f(z) = \bigcup(z^{\downarrow} \cap T_1)$ is a continuous surjection. Also f is 1-to-1 on $bT_2 \cap bT_1$ (some branches may remain maximal)

If we have an $\mathcal{A}_{T_1}$ with $T_1 = 2^{<\omega_1}$ and we enlarge the model, we just extend to $\mathcal{A}_{T_2}$ ($T_2 = 2^{<\omega_1}$) and (ground model) X_{T_1} is a subspace of X_{T_2} (robustness)

Theorem (Koszmider)

It is consistent to have $T_2 = 2^{<\omega_1+\omega}$ and $T_1 = 2^{<\omega_1}$ (and always using $Part(\mathcal{A}_t, t)$) such that

1. *X_{T_2} is first countable (remarkable) with continuous image*
2. *X_{T_1} has no points of countable character (but it is Frechet)*

a few more – the "shop" part of workshop

Theorem (with Shelah)

$\mathfrak{b} = \mathfrak{c}$ *implies there is an Efimov T-algebra (with* $T \subset 2^{<\mathfrak{c}}$*)*

Theorem

It is consistent to have T-algebra with $T = 2^{<\omega_1}$ *with a countable* $\{z_n : n \in \omega\} \subset X_T$ *that has no converging subsequences and yet* $\overline{\{z_n : n \in \omega\}} \setminus \{z_n : n \in \omega\}$ *is Frechet. It follows that* $\overline{\{z_n\}_n}$ *is a Moore-Mrowka space for a new reason. Also a countable union of closed Frechet subspaces that is not Frechet.*

Theorem

There is no Efimov space with weight less than $\mathfrak{s}$ *(the splitting number) but it is consistent with* $\mathfrak{s} > \omega_1$ *that there is an Efimov T-algebra with* $T = 2^{<\omega_1}$ *and so character* $< \mathfrak{s}$*.*

and more

Theorem (Juhasz, Kosmider, Soukup)

It is consistent that there is an T-algebra space X_T that is ω_1-free but not first-countable. [*the Juhasz dichotomy fails*]

also example of: 1st ctble init. ω_1-cpt $\not\rightarrow$ cpt

Theorem

It is consistent with Martin's Axiom and not CH to have an T-algebra such that X_T is Moore-Mrowka.

Corollary

(Koszmider and) I hope you will try using T-algebras.

Forcing, Stone spaces, and converging sequences II

Forcing methods for no free spaces

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Posets of countable elementary submodels

Fix a regular cardinal $\kappa \geq \mathfrak{c}^+$ and let $H(\kappa)$ denote the set of all sets whose transitive closure has cardinality κ. For most things, we just view $H(\kappa)$ as our model of set-theory.

Definition

The set $\mathbb{M}$ is the set of all finite $\in$-chains $\mathcal{M} = \{M_i : i < \ell\}$ of countable elementary submodels of $H(\kappa)$ (by default $M_i \in M_{i+1}$)

Facts

Let $\{M_i : i < \ell\} \in \mathbb{M}$ and let $i < j < \ell$ and $\emptyset \neq S \in M_i$:

1. $M_i \in M_j$ and $M_i \subset M_j$; $S \cap M_i \neq \emptyset$,
2. $M_i \cap \omega_1 = \delta_i \in M_j \cap \omega_1$, and
3. if $S \setminus M_i \neq \emptyset$ then S is uncountable and $S \cap M_j \setminus M_i$ is not empty ($\emptyset \neq S \setminus M_i \in M_j$).

$\mathbb{M}$-type posets for using PFA

Definition (elementary submodels as side conditions)

A poset $\mathbb{P}$ will be an $\mathbb{M}$-type poset if $p \in \mathbb{P}$ implies

1. p is a function with domain $\mathcal{M}_p \in \mathbb{M}$,
2. $p(M_i) \in M_j$ for $M_i \in M_j$ and $M_i, M_j \in \mathcal{M}_p$
3. for each $M \in \mathcal{M}_p$, $\quad p < (p \cap M) = p \restriction (\mathcal{M}_p \cap M)$
4. $p < q$ implies $p \supset q \quad$ (and $\mathcal{M}_p \supset \mathcal{M}_q$)

We may impose extra conditions on p being in $\mathbb{P}$ and $p < q$

Fact (elementary submodels can sometimes squeeze in between)

Suppose $\kappa < \theta$, $|\bar{M}| = \aleph_0$, $\kappa \in \bar{M} \prec H(\theta)$ and $\mathcal{M} \in \mathbb{M}$, then

1. if $\bar{M}_\kappa = \bar{M} \cap H(\kappa)$ then $\{\bar{M}_\kappa\} \in \mathbb{M}$,
2. if $\bar{M}_\kappa \in \mathcal{M}$, then there exists M' such that (?for any? M')
$$\mathcal{M} \cup \{M'\} <_{\mathbb{M}} \mathcal{M} \text{ and } \mathcal{M} \cap \bar{M}_k \in M' \in \bar{M}_\kappa$$

PFA and $\mathbb{M}$-type posets

Definition

$\mathbb{M}$-type poset $\mathbb{P}$ is **proper** if for countable $\bar{M} \prec H(\theta)$ and $\mathbb{P} \in \bar{M}$,

if $\bar{M}_\kappa \in \mathcal{M}_p$ for some $p \in D$, $D \in \bar{M}$, then there is an $M' \in \bar{M}$ and a $d \in \bar{M}$ such that (compatible mirror image in $\bar{M}$)

1. $\mathcal{M}_p \cap \bar{M}_\kappa \in M'$, $\{M'\} \in \mathbb{M}$ (M' is a mirror of $\bar{M}_\kappa$)
2. $\mathcal{M}_d \cap M' = \mathcal{M}_p \cap M' = \mathcal{M}_p \cap \bar{M}_\kappa$ and $d \in D \cap \mathbb{P}$
3. $|\mathcal{M}_p \setminus \bar{M}| = |\mathcal{M}_d \setminus M'|$ ((1)-(3) $\equiv$ d is mirror of p)
4. $d \cup p \in \mathbb{P}$ and is an extension! of both d and p

only (4) requires work

PFA states that if $\mathbb{P}$ is proper, then any ω_1 sized set $\mathcal{D}$ of dense subsets of $\mathbb{P}$ has a $\mathcal{D}$-generic filter G (finitely $\downarrow$-directed).

i.e. MA(ω_1) for proper posets

$G \cap D \neq \emptyset$ imposes the D-described property on most $p \in G$

The poset $\mathbb{M}$ itself (i.e. identity functions) and the poset $\mathbb{M}_1$ (where $p(M) \in \omega_1$) are proper.

(elementary!) Proof that $\mathbb{M}_1$ is proper (it is not ccc)

Let $\bar{M} \prec H(\theta)$, $D \in \bar{M}$ and $\bar{M}_\kappa \in \mathcal{M}_p$ for some $p \in D \cap \mathbb{M}_1$.

Since $(p \cap \bar{M}) \in \bar{M} \cap H(\kappa)$, we may choose $M' \in \bar{M}_\kappa$ such that $(p \cap \bar{M}) \in M' \prec H(\kappa)$. $(p \cap \bar{M})$ doesn't really mention p

By simple elementarity, there is mirror $d \in \bar{M}$ of p. In this poset,
no extra requirements on $\in \mathbb{P}$ and $<_{\mathbb{P}}$, so
we get $d \cup p \in \mathbb{M}_1$ is the desired extension

each $\bar{M}_\kappa \in \mathcal{M}_p$ for some $p \in \mathbb{P}$
is needed to ensure that G will be uncountable

what do we get?

If $\mathcal{D}$ is a family of ω_1 many dense subsets of a proper $\mathbb{M}$-type poset $\mathbb{P}$, and if G is $\mathcal{D}$-generic,

then let $X_G = \{p(M) : p \in G \ \& \ M \in \mathcal{M}_p\}$ (can it be useful?)

It's not obvious, but $\mathcal{D}$ can always be chosen so as to ensure that
$C_G = \{M \cap \omega_1 : (\exists p \in G) \ \ M \in \mathcal{M}_p\}$ is a c.u.b. in ω_1

which M are successors?
and which equal union of predecessors

Corollary (PFA)

For any family $\{a_\alpha : \alpha \in \omega_1\}$ of infinite subsets of ω_1, there is a cub C_G (from $\mathbb{M}_1$) that mod finite contains no a_α.

Theorem (PFA - Todorcevic)

There are no S-space topologies on ω_1.

Proof.

Assume $\alpha \in U_\alpha$ (open) and $\overline{U_\alpha} \subset \alpha{+}1$ (i.e. regular non-Lindelof)

$p \in \mathbb{P}$ providing $p : \mathcal{M}_p \to \omega_1$

and $M_1 \in M_2$ implies $p(M_1) \notin U_{p(M_2)}$ extra condition on $\mathbb{P}$

will ensure that X_G uncountable discrete

no extra condition on $p < q$ □

after the next proof you'll know how to find a good mirror d so that $d \cup p \in \mathbb{P}$ and in this case $d \cup p < d, p$ by mirror

no extra on $<_{\mathbb{P}}$

Let Y be a countably compact subspace of a regular space X of countable tightness. Suppose that $\mathcal{F}$ is a maximal free filter of closed subsets of Y. ($\mathfrak{p} = \mathfrak{c}$ implies Y exists if X is Moore-Mrowka)

Let $\mathcal{U}$ be the family of open subsets of X whose closures miss the closure of a member of $\mathcal{F}$ (or a nbd assignment as in S-space)

Definition

$p \in \mathbb{P}(\mathcal{F})$ ($\mathbb{M}$-like) providing $p(M^p_\delta) = (x^p_\delta, U^p_\delta)$ where $\delta = M^p_\delta \cap \omega_1$ and $x^p_\delta \in U^p_\delta \in \mathcal{U}$ and $Y \ni x^p_\delta \in \overline{F \cap M^p_\delta}$ for all $F \in \mathcal{F} \cap M^p_\delta$.

Let $W^p_\delta = U^p_\delta \cap \bigcap\{U^p_\gamma : x^p_\delta \in U^p_\gamma\}$. just a little transitivity twist

for $p < q$ we have extra demand that $x^p_\alpha \in W^q_\delta$ (squeeze condition) if $M^p_\alpha \notin \mathcal{M}_q$ and δ is minimal such that $\alpha \in M^q_\delta$

challenge: how to get mirror d to respect W^p_δ?

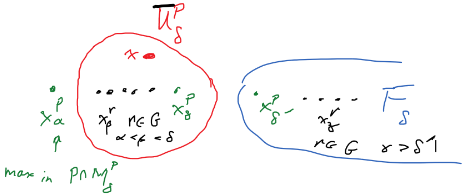
Lemma (PFA)

There are ω_1-many dense sets so that $X_G = \{(x^p_\delta, U^p_\delta) : p \in G\}$ is (essentially an) uncountable free sequence (in X!). If X has character at most ω_1, this is homeomorphic to $C_G \approx \omega_1$.

Proof.

Fix any $x \in X$ and suppose there is a minimal $\delta \in C_G$ so that $x \in \overline{\{x_\alpha : \alpha \in C_G \cap \delta\}}$. Let $\delta' = \min(C_G \setminus \delta{+}1)$ (as we may).

Choose any $p \in G$ such that $x_\delta = x^p_\delta$ and $x_{\delta'} = x^p_{\delta'}$. Choose any $F_\delta \in M^p_{\delta'} \cap \mathcal{F}$ so that U^p_δ and F_δ have disjoint closures. □



$\mathbb{P}(\mathcal{F})$ is proper

Let $\bar{M}, \bar{M}_\kappa, D, p, M'$ be as usual.
i.e. $p \in D$, $D \in \bar{M}$, $\{\mathcal{U}, \mathcal{F}, \mathbb{P}(\mathcal{F})\} \in \bar{M}$, and $p \cap \bar{M} \in M' \in \bar{M}_\kappa$

Let $\delta_0 = \bar{M} \cap \omega_1$, and $\mathcal{M}_p \setminus M' = \{M^p_{\delta_i} : i < \ell\}$ and $t_p \in Y^{\leq \ell}$ (viewed as a tree) where $t_p(i) = x^p_{\delta_i}$. Let $W = W^p_{\delta_0}$ – hard part

Let $D' = \{q \in D \cap \mathbb{P}(\mathcal{F}) : q \cap M' = p \cap M', \ |\mathcal{M}_q \setminus M'| = \ell\}$
set of mirrors

Let $T = \{t_q \restriction j : j \leq \ell, q \in D'\} \subset Y^{\leq \ell}$ where t_q is defined just as we did t_p. Note that $T \in \bar{M}_\kappa$, $D' \notin \bar{M}_\kappa$

Magic Mirror 1: for $i < \ell$, $t_p \restriction i \in M^p_{\delta_i}$ and $t_p(i) = p(M^p_{\delta_i}) \notin M^p_{\delta_i}$ and so $t_p \restriction i$ has $\mathcal{F}^+$ many immediate successors in T.

Magic Mirror 2: T has a subtree $\{t_p \restriction i : i < \ell\} \subset T' \in \bar{M}_\kappa$ such that every $t \in T' \cap Y^{<\ell}$ has $\mathcal{F}^+$ many immediate successors in T'.

this is the cool part

We use

Magic Mirror 2: T has a subtree $\emptyset \in T' \in \bar{M}_\kappa$ and every $t \in T' \cap Y^{<\ell}$ has $\mathcal{F}^+$ many immediate successors in T'.

(we walked down from t_p)

to finish the proof. we walk back up from $t_0 = \emptyset \in T'$ to pick d

$W \notin \bar{M}$ is a neighborhood of x_0^p and using countable tightness

x_0^p is in the closure of every $H \cap \bar{M}_\kappa$, $H \in \mathcal{F}^+ \cap \bar{M}_\kappa$

$\mathcal{F}^+$ many choices

For $i < \ell$, choose $t_{i+1} = t_i^\frown y_i \in T' \cap \bar{M}_\kappa$ so that $y_i \in W$.

Since $t_\ell \in T$, it follows that there is a $d \in D' \cap \bar{M}_\kappa$ such that $t_\ell = t_d$ and ($t_d \subset W$ so) $d \cup p$ is an extension of p, d as required.

Luzin gaps and ω_1-sequences

Definition

An adf $\mathcal{A} = \{a_\alpha : \alpha \in C \subset \omega_1\} \subset [\omega]^{\aleph_0}$ is Luzin if

(Luzin condition) $(\forall\alpha)(\forall n)\{\beta < \alpha : a_\beta \cap a_\alpha \subset n\}$ is finite

no splitting a gap

If $\{\alpha : a_\alpha \subset^* Y\}$ is uncountable, then

the set $\{\alpha : a_\alpha \cap Y =^* \emptyset\}$ is countable

tool for not ω_1-free

If $a_\alpha \to x_\alpha \in X$ (compact T_2) for all α, why would $\mathcal{A}$ do this?

then $\{x_\alpha : \alpha \in \omega_1\}$ is a converging ω_1-sequence $\mathbb{M}$-type forcing!

forcing a Luzin family

Theorem (PFA)

If ω is dense in a compact sequential space X, then every $x \in X \setminus \omega^{(2)}$ is the limit of a converging ω_1 sequence from $\omega^{(1)}$.

causing a big challenge to $\omega^{(2)}$

Proof.

Let $\mathcal{U}$ be an ultrafilter on ω such that x is the $\mathcal{U}$-limit.

$p \in \mathbb{P}$ ($\mathbb{M}$-type) if $\quad X, \mathcal{U} \in M \in \mathcal{M}_p$

1. $p(M) = a_p^M \in [\omega]^{\aleph_0}$ is a converging sequence in X (to x_M^p)
2. $a_M^p \subset^* U$ for all $U \in M \cap \mathcal{U}$ $\quad$ (automatically distinct x_M^p)
3. for $p < q$ and $M \in \mathcal{M}_p \setminus \mathcal{M}_q$ and $M \in M' \in \mathcal{M}_q$, $a_M^p \cap a_{M'}^q$ is not contained in $|\mathcal{M}_q|$ $\quad$ (this yields Luzin)

□

The previous theorem gave some PFA limitations on the structure of compact sequential spaces with infinite sequential order.

it also had an application to a question raised by Eisworth related to the Moore-Mrowka problem after independence with CH proof

and another new application

Theorem (with K.P. Hart)

PFA implies that The Juhasz dichotomy holds:

Every ω_1-free compact space is first countable.

what about L-spaces?

Theorem (PFA - Szentmiklossy)

There are no first-countable L-space topologies on ω_1.

Proof.

Assume $\alpha \in U_\alpha(n)$ and $\overline{U}_\alpha(n) \cap \alpha = \emptyset$ (1st ctble, not separable)
$p \in \mathbb{P} \subset \mathbb{M}$ providing $M_1 \in M_2$ implies $\delta_{M_2} \notin U_{\delta_{M_1}}$
not proper if we use J. Moore's L-space

Let $D \in \bar{M}$ and $\bar{M}_\kappa \in \mathcal{M}_p \in D$, Pick $p \cap \bar{M} \in M' \in \bar{M}_\kappa$ as usual.
Let $\ell = |\mathcal{M}_p \setminus \bar{M}|$ and define usual set of mirrors

$$D' = \{q \in D \cap \mathbb{P} : q \cap M' = p \cap M',\ |\mathcal{M}_q \setminus M'| = \ell\} \in \bar{M}_\kappa$$

for $q \in D'$, let $K_q = \{\delta_M : M \in \mathcal{M}_q \setminus M'\}$

Let $K_p \in \mathcal{L} = \mathcal{K} = \{K_q : q \in D'\}$. $\mathcal{L}$ is unbounded where

$\mathcal{L}' \subset \mathcal{L}$ is unbounded if $\{\min(L) : L \in \mathcal{L}'\}$ is uncountable $\square$

For each $L \in \mathcal{L}$, let $V(L) = \bigcup\{U_\delta : \delta \in L\}$,

If we can find $d \in D' \cap \bar{M}_\kappa$ so that $V(K_d) \cap K_p$ is empty, then $d \cup p$ shows that $\mathbb{P}$ is proper and our space is not hered. Lindelof

otherwise

For $\mathcal{L}' \subset \mathcal{L}$, let $V[\mathcal{L}'] = \bigcup\{V(L) : L \in \mathcal{L}'\}$ Choose unbounded $\mathcal{L}_1 \subset \mathcal{L}$ in $\bar{M}_\kappa$ so that $|V[\mathcal{L}_1] \cap K_p| > 0$ is **minimal** among unbounded $\mathcal{L}' \in \bar{M}_\kappa$ now more mirrors

$\mathcal{K}_\delta(\mathcal{L}_1)$ is the unbdd set of all K_d ($\delta \le \min(K_d)$) for $d \in D'$ st $\mathcal{L}_1 \in \min(\mathcal{M}_d \setminus M')$ and $|V[\mathcal{L}_1] \cap K_d|$ is min'l among unbdd $\mathcal{L}' \in \min(\mathcal{M}_d \setminus M')$

Fact: $V[\mathcal{L}_1] \cap \bigcup \mathcal{K}(\mathcal{L}_1)$ is not Lindelof.

For any $\alpha \in \bigcup \mathcal{K}(\mathcal{L}_1)$,

$$U_\alpha(n) \cap (V[\mathcal{L}_1] \cap \bigcup \mathcal{K}(\mathcal{L}_1)) \quad \text{is bounded (misses some } \mathcal{K}_\delta(\mathcal{L}_1))$$

so long as $\mathcal{L}(\alpha, n) = \{L \in \mathcal{L}_1 : U_\alpha(n) \cap V[L] = \emptyset\}$ is unbounded.