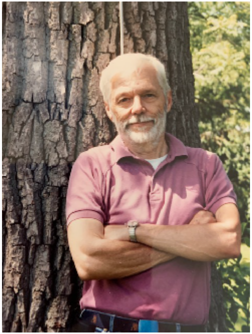
# W. Charles Holland and Automorphism Groups of Ordered Sets

Manfred Droste (Leipzig)



## Charles Holland (1935 - 2020)

Studies at Tulane University
1961 PhD on "*Extensions of Ordered Algebraic Structures*", supervisor: Paul Conrad

NATO Postdoctoral Fellowship for stay in Tübingen, Germany, with Helmut Wielandt

Positions:

- University of Chicago
- University of Wisconsin – Madison
- Bowling Green State University, Chair (1972 - 2002)

## Charles Holland (1935 - 2020)

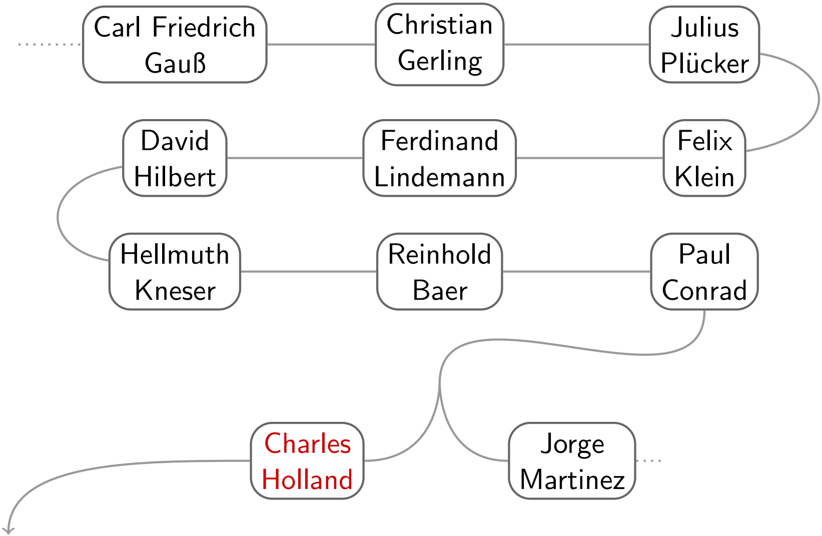
Hired for developing the doctoral program in Mathematics
Also: built an algebra group.

Center for Ordered Groups
Members: Andrew Glass, Steve McCleary, Warren McGovern, ...

Chairman of Department of Mathematics and Statistics
1984 Honorary Fellow of the Societe Francaise de l'Algebre Orderee
1993 Distinguished Research Professor by University's Board of Trustees

Member of "*The Logarythms*" (barbershop quartet)
Married to Claudia Holland (5 children)

# Scientific ancestry



# Scientific descendants

21 PhD Students:

Hollister, Herbert
McCleary, Stephen
Madell, Robert
Pierce, Keith
Scrimger, Jr., Edward
Glass, Andrew
Read, John
Ball, Richard
Roll, Janet
Smith, Jo
Franchello, James

Putt, Harold
Bardwell, Maureen
Feil, Todd
Droste, Manfred
Maroli, John
Wojciechowski, Piotr
Bobek, Ludmila
Dong, Yi
Button, Michael
Lafuente Rodriguez, Ramiro

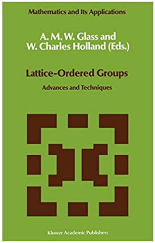
# Invited lectures & Conferences

Many invited lectures, including:
United States, Canada, Mexico, Czechoslovakia, France, Germany, Italy, former Soviet Union, China

Conference Organization (selection):

- 1984 Bowling Green with Andrew Glass
- 1991 Florida, for Paul Conrad's 70th birthday, with Jorge Martinez
- 1993 Luminy, with Peter Neumann
- 1995 Curacao, with Jorge Martinez
- 1997 Irvine, on Mathematical Psychology, Theory of Measurement, with Duncan Luce
- 1998 Nanjing, China

# Books edited




A.M.W. Glass, C. Holland:

Lattice-Ordered Groups. Advances and Techniques.

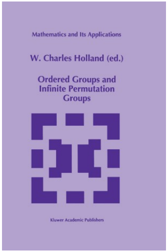
Kluwer Academic Publishers, 1989.


J. Martinez, C. Holland:

Ordered Algebraic Structures. The 1991 Conrad conference, Univ. of Florida, Gainesville, USA, dedicated to Paul F. Conrad on the occasion of his 70th birthday.

Kluwer Academic Publishers, 1993.

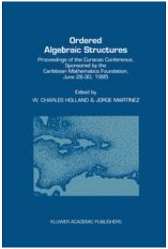
# Books edited




C. Holland:

Ordered Groups and Infinite Permutation Groups. Conference in Luminy, France, 1993.

Kluwer Academic Publishers, 1996.

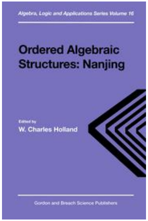



C. Holland, J. Martinez:

Ordered Algebraic Structures. Curaçao conference, Netherlands Antilles, 1995.

Kluwer Academic Publishers, 1997.

# Books edited

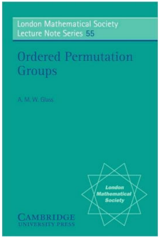



C. Holland:

Ordered Algebraic Structures. Conference, Nanjing, China, 1998.

Gordon and Breach Science Publishers, 2001.

Book:




A.M.W. Glass: Ordered Permutation Groups,

London Mathematical Socety Lecture Note Series vol. 55,

Cambridge University Press, 1981.

1901-... abelian ordered groups Hölder, Hahn

1930-... (abelian) lattice-ordered groups Birkhoff

1955/60-... non-abelian $\ell$-groups P. Conrad

Example. $(\Omega, \leq)$ linearly ordered set (chain)

$A(\Omega) := \mathrm{Aut}(\Omega, \leq)$

$= \{$all order-preserving permutations of $(\Omega, \leq)\}$

$f, g \in A(\Omega) \leadsto f \leq g \ :\Leftrightarrow\ \forall x \in \Omega: xf \leq xg$

$\Rightarrow (A(\Omega), \leq, \cdot)$ is an $\ell$-group

$$x(f \vee g) = \max\{xf, xg\}$$

$\leadsto$ infinite permutation groups G. Higman
H. Wielandt

Theorem 1 (Holland 1963).

Every $\ell$-group $\mathcal{G}$ can be $\ell$-embedded in $A(\Omega)$ for some chain $(\Omega, \leq)$.

Pf idea: Construct $(\Omega, \leq)$ from values $V_g$ of $g \in \mathcal{G}$.

$(\Omega, \leq)$ is <u>doubly homogeneous</u>

$:\Leftrightarrow \quad \forall\, a<b,\ c<d$ in $\Omega$ $\exists\, g \in A(\Omega)$:

$$ag = c, \quad bg = d$$

<u>Example</u> $(\mathbb{Q}, \leq)$, $(\mathbb{R}, \leq)$, all linearly ordered fields

<u>Prop. 1</u> (Weinberg). $(\Lambda, \leq)$ a chain

$\Rightarrow$ $\exists$ doubly homogeneous chain $(\Omega, \leq)$:

$\exists$ $l$-embedding: $A(\Lambda) \hookrightarrow A(\Omega)$.

<u>Pf.</u> 1) $\Omega = \mathbb{Q}(F(\Lambda))$ ordered field

2) $\Lambda \equiv \Lambda \times \{1\} \subseteq \Lambda \times \{1,2\}$ 

"fill up the gaps" $\rightsquigarrow$ $(\Omega, \leq)$ 

with $\Lambda \times \{1,2\} \subseteq \bar{\Omega} \setminus \Omega$,

$(\Omega, \leq)$ doubly homog. (Shelah + MD)

$\Rightarrow$ $A(\Lambda) \hookrightarrow A(\Omega)$ (Holland). $\square$

So: In Thm. A, $(\Omega, \leq)$ can be chosen doubly homogeneous.

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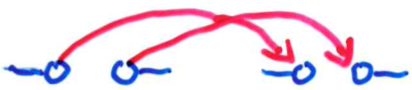
<u>Pf.</u> 1) $\Omega = \mathbb{Q}(F(\Lambda))$ ordered field

2) $\Lambda \cong \Lambda \times \{1\} \subseteq \Lambda \times \{1, 2\}$



"fill up the gaps" $\rightsquigarrow$ $(\Omega, \leq)$

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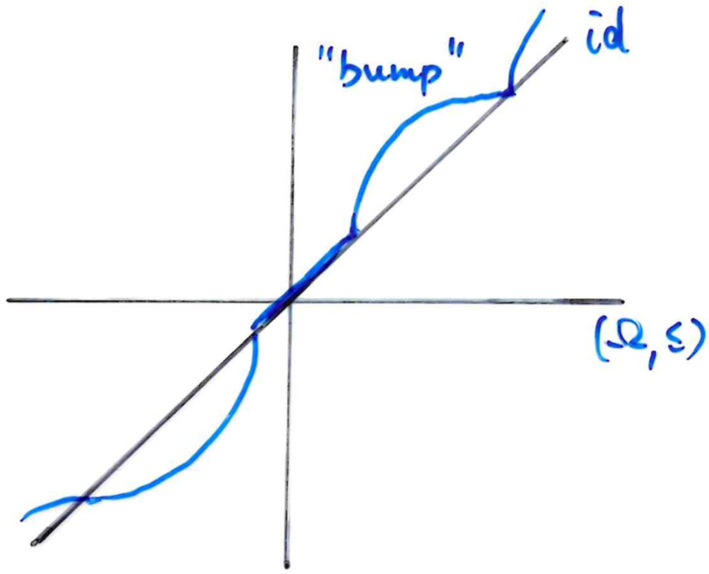
Advantage:
pictures




Prop. 2 (Holland '63). $(\Omega, \leq)$ doubly homog.

$f, g \in A(\Omega)$ are conjugate

$\Leftrightarrow$ the "bump structures" of $f, g$ are conjugate.

Cor. (Holland '63). Every $\ell$-group $\mathcal{G}$ can be embedded in a divisible $\ell$-group in which each element is a commutator.

Pf. Thm. 1 + Prop. 1 $\Rightarrow$ $\mathcal{G} \hookrightarrow A(\Omega)$, $\Omega$ doubly hom

$f \in A(\Omega)$, $n \in \mathbb{N}$ $\underset{\text{Prop. 2}}{\Rightarrow}$ $f = (f^n)^h$ for some $h \in A(\Omega)$

$$= (f^h)^n$$

$n = 2 \Rightarrow f^2 = f^g = g^{-1} f g$

$$f = f^{-1} g^{-1} f g = [f, g]. \qquad \square$$

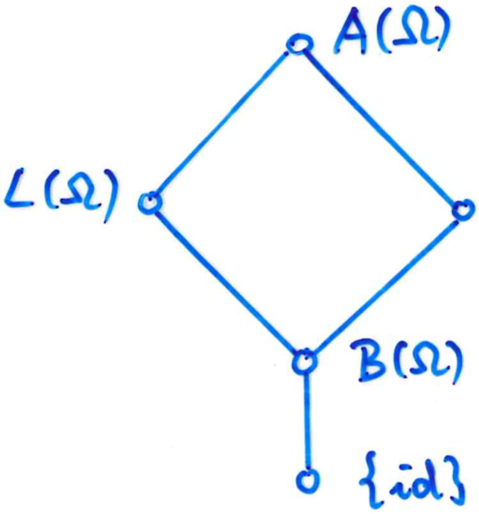
(Cf. Ore 1951 for $\mathrm{Sym}(\Omega)$.)

# Normal subgroups of $A(\Omega)$ $\quad (\Omega, \leq)$ doubly hom.

$R(\Omega) = \{ g \in A(\Omega) : \ \exists x \in \Omega \ \ \forall y \leq x : \ \ yg = y \}$

$L(\Omega) = \{ g \in A(\Omega) : \ \exists x \in \Omega \ \ \forall y \geq x : \ \ yg = y \}$

$B(\Omega) = L(\Omega) \cap R(\Omega)$ $\qquad$ bounded support




simple (Higman '54).

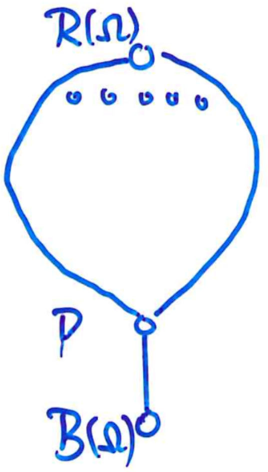
Thm. 2. $(\Omega, \leq)$ doubly homogeneous.

a) (Holland '63, Lloyd). $\quad \mathrm{cof}(\Omega) = \omega \quad \begin{pmatrix} \text{unbdd.} \\ \mathbb{Z}\text{-sequ.} \end{pmatrix}$

$\Rightarrow$ $R(\Omega)$, $L(\Omega)$, $B(\Omega)$ are <u>all</u> non-trivial proper normal subgroups of $A(\Omega)$.

e.g. $\Omega = \mathbb{Q}, \mathbb{R}$.

b) (R. Ball + MD '85) $\quad k = \mathrm{cof}(\Omega) \neq \omega \Rightarrow$




$\exists$ antichain of $2^{2^k}$ maximal proper normal subgroups of $R(\Omega)$.

Full description of normal subgroup lattice of $A(\Omega)$ in Shelah + MD '85.

Cor. (Holland '63). Every $\ell$-group $\mathcal{G}$ can be $\ell$-embedded into a simple $\ell$-group.

Pf. $\mathcal{G} \hookrightarrow B(\Omega)$, $\Omega$ doubly homog. $\quad \square$

## Structure theory

$(\mathcal{G}, \Omega)$ $\ell$-subgroup, o-2-transitve

$\Rightarrow$ $(\mathcal{G}, \Omega)$ o-primitive

(no non-trivial $\mathcal{G}$-congruences)

Holland '65:

$A(\Omega)$ transitive + o-primitive $\Rightarrow$

either $A(\Omega)$ o-2-transitive

or $\underset{\text{subgroup}}{(\Omega, \leq) \cong \mathbb{R}}$, $(A(\Omega), \Omega)$ right regular repr.

Example. $z : \mathbb{R} \to \mathbb{R}$ translation
$x \mapsto x+1$

$$\mathcal{G} := C_{A(\mathbb{R})}(z) = \{ g \in A(\mathbb{R}) : \ zg = gz \}$$

is transitive + o-primitive on $\mathbb{R}$,
not o-2-trans.

## Thm. 3 (McCleary '67).

$(\mathcal{G}, \Omega)$ trans. o-primitive $\ell$-permutation group.

$\Rightarrow$ either

a) $(\mathcal{G}, \Omega)$ is o-2-transitive, or

b) $(\Omega, \leq) \underset{\text{subgroup}}{\subseteq} \mathbb{R}$, $(\mathcal{G}, \Omega)$ right regular repres., or

c) $(\mathcal{G}, \Omega)$ is periodic, i.e.

$\mathcal{G} = C_{A(\Omega)}(z)$ for some unbounded $z \in A(\bar{\Omega})$.

McCleary '76: ... without transitivity assumption

Thus: o-primitive "components" described ✓

$\rightsquigarrow$ reconstruct general $(\mathcal{G}, \Omega)$

$\rightsquigarrow$ generalized wreath products

Holland '69, Holland + McCleary '69.

# Varieties

**Thm. 4** (Holland '76). $(\mathcal{G}, \Omega)$ $\ell$-perm. group, o-2-trans. or periodic

$\mathcal{V} :=$ variety of $\ell$-groups generated by $\mathcal{G}$

$\Rightarrow$ $\mathcal{V} = \{$all $\ell$-groups$\}$.

Much further work on varieties

... mv-algebras, $\ell$-groups with unit

**Thm. 5** (Holland + MD '07). $\quad u: \mathbb{R} \to \mathbb{R}$, $x \mapsto x+1$

$BAut(\mathbb{R}) := \{g \in A(\mathbb{R}) : \exists n \in \mathbb{N}.\ |g| \leq u^n\}$

has an antichain of $2^{2^{\aleph_0}}$ normal subgroups.

**Thm. 6.** a) (Holland + McCleary '79).

Any free $\ell$-group has solvable word problem.

b) (McCleary '85).

Any free $\ell$-group $F$ on $\geq 2$ generators has a representation as o-2-trans. $\ell$-perm. group $(F, \Omega)$.

# $Th(A(\Omega))$ and $Aut(A(\Omega))$

**Thm. 8** (Gurevich + Holland '81).

$A(\Omega)$ 1-transitive on $(\Omega, \leq)$

a) $A(\Omega) \equiv A(\mathbb{R})$ as $\ell$-groups

$\Rightarrow (\Omega, \leq) \cong (\mathbb{R}, \leq)$.

b) $A(\Omega) \equiv A(\mathbb{Q})$

$\Rightarrow (\Omega, \leq) \cong (\mathbb{Q}, \leq)$ or $(\mathbb{R} \setminus \mathbb{Q}, \leq)$.

**Prop. 3.**

$(\Omega, \leq)$, $(\Lambda, \leq)$ doubly homog. chains,

$\psi : A(\Omega) \to A(\Lambda)$ group isomorphism

$\Rightarrow \exists$ isom. or anti isom. $\varphi : \Omega \to \Lambda'$ <u>orbit</u> of $A(\Lambda)$ in $\bar{\Lambda}$

s.th. $\varphi$ induces $\psi$, i.e.

$$\begin{tikzcd}
\Omega \arrow[r, "g"] \arrow[d, "\varphi"'] & \Omega \arrow[d, "\varphi"] \\
\Lambda' \arrow[r, "g^\psi"] & \Lambda'
\end{tikzcd}
\qquad
\begin{array}{c}
g \in A(\Omega) \\
\downarrow \psi \downarrow \\
g^\psi \in A(\Lambda) \subseteq A(\bar{\Lambda})
\end{array}$$

Thm. 9 (Holland '74, McCleary '73).

$\exists$ doubly homog. chain $(\Omega, \leq)$ s.th.

- $\mathrm{Aut}(A(\Omega)) = \mathrm{Inn}(A(\Omega))$
- $\mathrm{Out}(A(\Omega)) := \mathrm{Aut}(A(\Omega)) / \mathrm{Inn}(A(\Omega)) \cong \begin{cases} \mathbb{Z}_2 \\ V_4 \end{cases}$

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**Thm. 10** (Shelah + MD '02).

$\mathcal{G}$ any group

$\Rightarrow$ $\exists$ doubly homog. chain $(\Omega, \leq)$ s.th.

$\mathrm{Out}(A(\Omega)) \cong \mathcal{G}$.

**Pf.** cf. Prop. 3 and Prop. 1.

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$\mathrm{Out}(A(\Omega)) \cong \mathcal{G}$.

Pf. cf. Prop. 3 and Prop. 1.

**Thm. 11** (Giraudet, Göbel + MD '01).

$\mathcal{G}$ any group

$\Rightarrow$ $\mathcal{G} \cong \mathrm{Out}(H)$ for some simple group $H$.

Moreover, $H \cong \mathrm{Aut}(\underbrace{\Omega, R}_{\text{homog. relational structure}})$, $R \subseteq \Omega^3$