Fraction-dense algebraic frames

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A frame is a complete lattice L in which the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

holds for all $a \in L$ and $S \subseteq L$.

A frame homomorphism $h\colon L \to M$ is a map between frames preserving finite meets and all joins.

An element a of L is compact if whenever $a \leq \bigvee T$ then $a \leq \bigvee S$ for some finite $S \subseteq T$. We denote by $\mathfrak{k}(L)$ the set of all compact elements of L.

An element of L is called a polar if it is of the form

$$a^{\perp} = \bigvee \{x \in L \mid x \wedge a = 0\}$$

for some $a \in L$. We write $\mathrm{Pol}(L)$ for the set of all polars of L.

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L is an algebraic frame if for every $a \in L$,

$$a = \bigvee \{x \in \mathfrak{k}(L) \mid x \leq a\}.$$

If L is an algebraic frame such that $a \wedge b \in \mathfrak{k}(L)$ for all $a, b \in \mathfrak{k}(L)$, then L is said to have the Finite Intersection Property, abbreviated FIP.

If A is a commutative ring with identity, we write RId(A) for the lattice of its radical ideals ideals. RId(A) is an algebraic frame with the FIP.

A frame homomorphism between algebraic frames is called a coherent map if it sends compact elements to compact elements.

FIPFrm denotes the category of algebraic frames with the FIP together with coherent maps.

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In the article

A.W. Hager and J. Martínez
Fraction-dense algebras and spaces
Can. J. Math. **45** (1993), 977–996,

the authors define an f-ring A to be fraction-dense if, in the language of ℓ-groups, the classical ring of quotients of A is rigid in the maximal ring of quotients of A.

Then they show that a reduced f-ring A is fraction-dense precisely when its annihilator ideals are principal polars, that is, are of the form $\mathrm{Ann}^2(a)$ for $a \in A$.

Lemma

If A is a reduced f-ring, then

$$\{\mathrm{Ann}^2(a) \mid a \in A\} = \{\mathrm{Ann}^2(I) \mid I \text{ is a finitely generated ideal of } A\}.$$

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For an arbitrary commutative ring R, denote by $\mathcal{A}(R)$ the set of annihilator ideals of R. Then

$$\mathcal{A}(R) = \{\mathrm{Ann}^2(x) \mid x \in R\} \quad \text{iff} \quad \mathcal{A}(R) = \{\mathrm{Ann}(y) \mid y \in R\}. \qquad (\dagger)$$

Proposition

A reduced f-ring A is fraction-dense iff for every $I \in \mathrm{RId}(A)$ there exists some $J \in \mathfrak{k}(\mathrm{RId}(A))$ such that $I^\perp = J^\perp$.

Definition

An algebraic frame L with the FIP is fraction-dense if for every $a \in L$ there exists $c \in \mathfrak{k}(L)$ such that $a^\perp = c^\perp$.

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Fraction-dense frames need not be compact, as the chain

$$L = \{0 < 1 < 2 < \cdots < \omega\}$$

shows.

Some Terminology

1. An element $a \in \mathfrak{k}(L)$ is called a:
   - unit if it is a dense (meaning that its polar is 0);
   - component if there is a $b \in \mathfrak{k}(L)$ such that $a \wedge b = 0$ and $a \vee b$ is dense.
2. We write $\mathrm{Com}(L)$ for the set of components of L.
3. If $\mathfrak{k}(L) = \mathrm{Com}(L)$, then L is said to be complemented.

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Proposition

The following are equivalent for an algebraic frame L with the FIP.

1. *L is fraction-dense.*
2. *For every $a \in L$, there exists $c \in \mathfrak{k}(L)$ such that $a^\perp = c^{\perp\perp}$.*
3. *For every $a \in L$, there exists $c \in \mathrm{Com}(L)$ such that $a^\perp = c^{\perp\perp}$.*
4. *For every $a \in L$, there exists $c \in \mathrm{Com}(L)$ such that $a^\perp = c^\perp$.*
5. $\mathrm{Pol}(L) = \{c^\perp \mid c \in \mathrm{Com}(L)\}$.

Using results from the article

P. Bhattacharjee
Two spaces of minimal primes
J. Alg. Appl. **11** (2012), Article 1250014,

we also have the characterization:

Corollary

An algebraic frame L with the FIP *is fraction-dense iff* $\mathrm{Min}(L)$ *is a Stone space.*

We also immediately deduce from proposition above the following result.

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Corollary

Every fraction-dense frame is complemented.

Recall that a Tychonoff space is basically disconnected if the closure of every cozero-set is open, and it is extremally disconnected if the closure of every open set is open.

Example

Let X be a basically disconnected space which is not extremally disconnected. Then $\mathrm{RId}(C(X))$ is complemented but not fraction-dense.

In the paper of Hager and Martínez cited earlier, they prove that:

> *An f-ring is fraction-dense iff its total ring of quotients is strongly projectable.*

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> *An f-ring is fraction-dense iff its total ring of quotients is strongly projectable.*

For an algebraic frame L with the FIP and possessing a unit, we set

$$Q(L) = \{p \in \mathrm{Pr}(L) \mid\ p \text{ is above no unit of } L\}.$$

Remark

A prime ideal of a reduced f-ring A belongs to $Q(\mathrm{RId}(A))$ if and only if it consists entirely of zero-divisors.

Using ideas developed by Martínez in

J. Martínez
Unit and kernel systems in algebraic frames
Algebra Universalis **55** (2006), 13–43,

we see that the map $q_L\colon L \to L$ given by

$$q_L(a) = \bigwedge\{p \in Q(L) \mid a \leq p\}$$

is a nucleus.

We shall write qL for the frame $\mathrm{Fix}(q_L)$ and $q_L\colon L \to qL$ for the induced frame homomorphism. When there is only one frame under consideration, we shall suppress the subscript.

By some general theory in paper of Martínez mentioned above, we have that:

(a) qL is an algebraic frame with the FIP and $q\colon L \to qL$ is a dense onto coherent map.

(b) For any $c \in \mathfrak{k}(L)$, $q(c) = 1$ if and only if c is a unit.

It is not hard to show that

$$\mathrm{Pr}(qL) = Q(L).$$

Recall that an algebraic frame L is called strongly complemented if $a^{\perp} \vee a^{\perp\perp} = 1$ for every $a \in L$.

Theorem

An algebraic frame L with the FIP and having units is fraction-dense iff qL is strongly projectable.

Recall that an algebraic frame L is called strongly complemented if $a^{\perp} \vee a^{\perp\perp} = 1$ for every $a \in L$.

Theorem

An algebraic frame L with the FIP and having units is fraction-dense iff qL is strongly projectable.

Recall that a frame homomorphism $h\colon L \to M$ is called skeletal if for any $a, b \in L$ with $a^{\perp} = b^{\perp}$, $h(a)^{\perp} = h(b)^{\perp}$.

Skeletal homomorphisms are precisely those that send dense elements to dense elements, and they include:

- all the dense onto ones;
- all of the form $a \wedge (-)\colon L \to {\downarrow}a$
- all of the form $a \vee (-)\colon L \to {\uparrow}a$, for $a = a^{\perp\perp}$.

Proposition

(a) If $h\colon L \to M$ is a onto skeletal coherent map and L is fraction-dense, then M is fraction-dense.

(b) If $h\colon L \to M$ is a dense onto coherent map, then L is fraction-dense iff M is fraction-dense. Hence, L is fraction-dense iff qL is fraction-dense.

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Proposition

Let $(L_i \mid i \in I)$ *be a family of algebraic frames with the* FIP*.*

(a) *If* $\prod_i L_i$ *is fraction-dense, then each* L_i *is fraction-dense.*

(b) *If* I *is finite and each* L_i *is fraction-dense, then* $\prod_i L_i$ *is fraction-dense.*

Example

For each $n \in \mathbb{N}$, let L_n be any non-trivial algebraic frame. Then $\prod_n L_n$ is not fraction-dense.

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Theorem

Let $h\colon L \to M$ be a coherent map between algebraic frames with the FIP *and having units. Then there exists a coherent map $\widetilde{h}\colon qL \to qM$ making the diagram*

$$
\begin{array}{ccc}
L & \xrightarrow{\quad h \quad} & M \\
{\scriptstyle q_L}\downarrow & & \downarrow{\scriptstyle q_M} \\
qL & \xrightarrow{\quad \widetilde{h} \quad} & qM
\end{array}
$$

commute iff h maps units to units. Furthermore, it is unique.

The *d*-nucleus on an algebraic frame L with the FIP was introduced by Martínez and Zenk in

J. Martínez and E.R. Zenk
When an algebraic frame is regular
Algebra Universalis **50** (2003), 231–257.

It is defined by

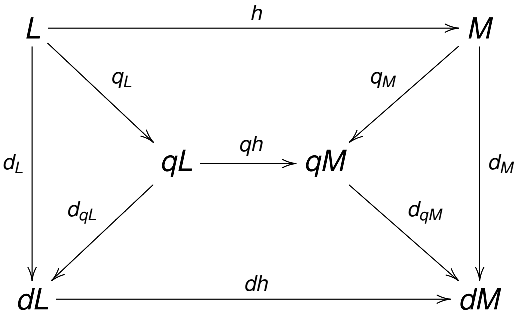
$$d(a) = \bigvee \{c^{\perp\perp} \mid c \in \mathfrak{K}(L) \text{ and } c \leq a\}.$$

The resulting quotient frame is denoted by dL. It is an algebraic frame with the FIP.

The mapping $d_L\colon L \to dL$ induced by d is a dense onto coherent map.

Theorem

If $h\colon L \to M$ is a skeletal map between algebraic frames with the FIP *and possessing units, then the diagram*



commutes.

Recall that skeletal means sending dense elements to dense elements. Weakening that, we formulate the following definition.

Definition

A coherent map $h\colon L \to M$ is weakly skeletal if it maps units to units.

Proposition

The following are equivalent for a coherent map $h\colon L \to M$ between algebraic frames with the FIP and possessing units.

1. *h is a weakly skeletal.*
2. $h_*[qM] \subseteq qL$.
3. $h_*[Q(M)] \subseteq Q(L)$.

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Let X be a Tychonoff space and denote by $\mathscr{C}_z(X)$ the lattice of z-ideals of $C(X)$. It is known that $\mathscr{C}_z(X)$ an algebraic frame with the FIP.

For each open set $U \subseteq X$ put

$$\boldsymbol{M}_U = \{f \in C(X) \mid \operatorname{coz}(f) \subseteq U\}.$$

It was shown by Martinez and Zenk that

$$\mathfrak{k}(\mathscr{C}_z(X)) = \{\boldsymbol{M}_U \mid U \text{ is a cozero-set of } X\}.$$

Example

Let X be an almost P-space which is not discrete and denote by X_d its descretization. It is not hard to see that the mapping

$$\phi\colon \mathfrak{k}(\mathscr{C}_z(X)) \to \mathfrak{k}(\mathscr{C}_z(X_d)) \qquad \text{given by} \qquad \boldsymbol{M}_U \mapsto \boldsymbol{M}_U^d$$

is a lattice homomorphism. Its extension $\widehat{\phi}\colon \mathscr{C}_z(X) \to \mathscr{C}_z(X_d)$ is a weakly skeletal coherent map which is not skeletal.

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