What are Weak Pseudo EMV-algebras?

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Boolean algebras, Generalized BA, MV-a

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- $\Gamma(G, u) = ([0, u]; \oplus, ^*, 0, u)$ is an MV-algebra,

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- $([0,b]; \oplus, \lambda_b, 0, b)$ - MV-algebra
- $\forall\, x \in M$, $\exists\, a \in \mathcal{I}(M)$ s.t. $x \leq a$

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- $(M; \oplus, ^-, 0, 1)$ - MV-algebra, then $(M; \vee, \wedge, \oplus, 0)$ - EMV-algebra with 1, $\lambda_1(x) = x^-$

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- EMV-algebras with top element are termwise equivalent to MV-algebras
- G- ℓ-group, G^+ not an EMV-algebra

- If we compare EMV-algebras with Wajsberg hoops, that are bottom-free subreducts of MV-algebras, we can find examples of Wajsberg hoops, which are not EMV-algebras and vice-versa.

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- Not every ideal of an MV-algebra is an EMV-algebra: Chang algebra has only one idempotent and it doesn't dominate all elements

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- $A \subseteq M$ is an *EMV-subalgebra* if A is closed under $\vee$, $\wedge$, $\oplus$ and 0
- $\forall\, b \in \mathcal{I}(M) \cap A$ *the set* $[0, b]_A := [0, b] \cap A$ *is a subalgebra of the* MV*-algebra* $([0, b]; \oplus, \lambda_b, 0, b)$

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- $x \in A$, there is $b \in A \cap \mathcal{I}(M)$ such that $x \leq b$
- $f : M_1 \to M_2$ is an *EMV-homomorphism* if preserves $\vee$, $\wedge$, $\oplus$ and 0, and for each $b \in \mathcal{I}(M_1)$ and for each $x \in [0, b]$, $f(\lambda_b(x)) = \lambda_{f(b)}(f(x))$

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qH: if $M \in \mathbb{V}$ and $h : M \to N$ is an EMV-homomorphism, then $h(M) \in \mathbb{V}$;

qS: if S is an EMV-subalgebra of $M \in \mathbb{V}$, then $S \in \mathbb{V}$;

qP: if $\{M_t\}$ is a system of EMV-algebras of $\mathbb{V}$, then $\prod_t M_t \in \mathbb{V}$,

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- **Theorem 0.4** *The system of q-subvarieties of* EMV*-algebras is a countably infinite lattice.*
- There is a one-to-one relationship between q-subvarieties of EMV-algebras and subvarieties of MV-algebras
- All q-subvarieties satisfy the same equational bases as do Di Nola-Lettieri for MV-varieties

Basic Representation of EMV-algebras

- **Theorem 0.5** *Every EMV-algebra* M *either has a top element or is topless and then there is a unique (up to isomorphism) EMV-algebra* N *with top element such that* M *can be embedded into* M*, the image of* M *is a maximal ideal of* N *and every element of* N *either belongs to the image of* M *or is a complement of some element from the image of* M*.*

Basic Representation of EMV-algebras

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- Generalization of the results of Conrad and Darnel for generalized Boolean algebras.

- Example: Let M be the set of all finite subsets of $\mathbb{N}$. Then M is topless and the representing EMV-algebra is the set of all finite and co-finite subsets of $\mathbb{N}$.

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- Every non-trivial EMV-algebra possesses a maximal ideal.

Categorical Equivalence

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- $\mathcal{PMV}$ - category - objects are couples (N, I), where N is an MV-algebra and I is a fixed maximal ideal of N having enough idempotent elements, $N = I \cup I'$ and no Boolean element of I is a top element of I.

- If (N_1, I_1) and (N_2, I_2) - two objects of $\mathcal{PMV}$-morphism in $\mathcal{PMV}$ from (N_1, I_1) into (N_2, I_2) is a homomorphism of MV-algebras $\phi : N_1 \to N_2$ such that $\phi(I_1) \subseteq I_2$.

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- $\Phi(\phi)(x) := \phi(x), \quad x \in I_1.$

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- Theorem: The functor Φ defines a categorical equivalence of the category $\mathcal{PMV}$ and the category of proper EMV-algebras $\mathcal{PEMV}$.

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- A mapping $s : M \to [0,1]$ is a *state-morphism* if s is an EMV-homomorphism from M into the EMV-algebra of the real interval $([0,1]; \vee, \wedge, \oplus, 0)$ with top element, where $\vee = \max$, $\wedge = \min$ and $\oplus$ is the truncated sum on $[0,1]$, i.e. $u \oplus v = \min\{u+v, 1\}$ ($u, v \in [0,1]$), such that there is an element $x \in M$ with $s(x) = 1$.

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- I-maximal ideal of M, then M/I is an EMV-subalgebra of $[0,1]$.

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- I-maximal ideal of M, then M/I is an EMV-subalgebra of $[0, 1]$.
- There is a one-to-one equivalence between maximal ideals and state-morphisms.
- $\mathcal{SM}(M) \neq \emptyset$ if $M \neq \{0\}$.

- Let M be a proper EMV-algebra and, for each $x \in M$, we put $x^* = \lambda_1(x)$. Given a state-morphism s on M, the mapping $\tilde{s} : N \to [0,1]$, defined by

$$\tilde{s}(x) = \begin{cases} s(x) & \text{if } x \in M, \\ 1 - s(x_0) & \text{if } x = x_0^*,\ x_0 \in M, \end{cases} \qquad x \in N,$$

is a state-morphism on N, and the mapping $s_\infty : N \to [0,1]$ defined by $s_\infty(x) = 0$ if $x \in M$ and $s_\infty(x) = 1$ if $x \notin M$, is a state-morphism on N.

- Moreover,
$\mathcal{SM}(N) = \{\tilde{s} \mid s \in \mathcal{SM}(M)\} \cup \{s_\infty\}$ and
$Ker(\tilde{s}) = Ker(s) \cup Ker_1^*(s)$, $s \in \mathcal{SM}(M)$,
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- A net $\{s_\alpha\}_\alpha$ of state-morphisms on M converges weakly to a state-morphism s on M if and only if $\{\tilde{s}_\alpha\}_\alpha$ converges weakly on N to $\tilde{s}$.

- Let M be an EMV-algebra. The mapping $\theta : \mathcal{SM}(M) \to \mathrm{MaxI}(M)$, defined by $s \mapsto Ker(s)$, is a homeomorphism. In addition, the following statements are equivalent

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- (i) M has a top element.

  (ii) $\mathcal{SM}(M)$ is compact in the weak topology of state-morphisms

  (iii) $\mathrm{MaxI}(M)$ is compact in the hull-kernel topology.

One-point compactification

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- M proper N its representation. Then $\mathcal{SM}(N)$ and $\mathrm{MaxI}(N)$ are the one-point compactifications of the spaces $\mathcal{SM}(M)$ and $\mathrm{MaxI}(M)$, respectively.

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(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = 0 \oplus x = x$;

(A3) $x \oplus 1 = 1 \oplus x = 1$;

(A4) $1^{\sim} = 0$; $1^{-} = 0$;

(A5) $(x^{-} \oplus y^{-})^{\sim} = (x^{\sim} \oplus y^{\sim})^{-}$;

(A6) $x \oplus (x^{\sim} \odot y) = y \oplus (y^{\sim} \odot x) = (x \odot y^{-}) \oplus y = (y \odot x^{-}) \oplus x$;

(A7) $x \odot (x^{-} \oplus y) = (x \oplus y^{\sim}) \odot y$;

(A8) $(x^{-})^{\sim} = x$.

- $x \leq y$ iff $x^{-} \oplus y = 1$

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- M – distributive lattice
- $x \vee y = x \oplus (x^{\sim} \odot y)$ and $x \wedge y = x \odot (x^{-} \oplus y)$.
- pseudo MV-algebra M is an MV-algebra iff $x \oplus y = y \oplus x$ for all $x, y \in M$.

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$(\Gamma(G, u); \oplus, ^{-}, ^{\sim}, 0, u)$ is a GMV-algebra.

- **Theorem 0.7** [Dvu 2002] *For any PMV-algebra* M*, there exists a unique* $($*up to isomorphism*$)$ *unital* ℓ*-group* G *with a strong unit* u *such that* $M \cong \Gamma(G, u)$.
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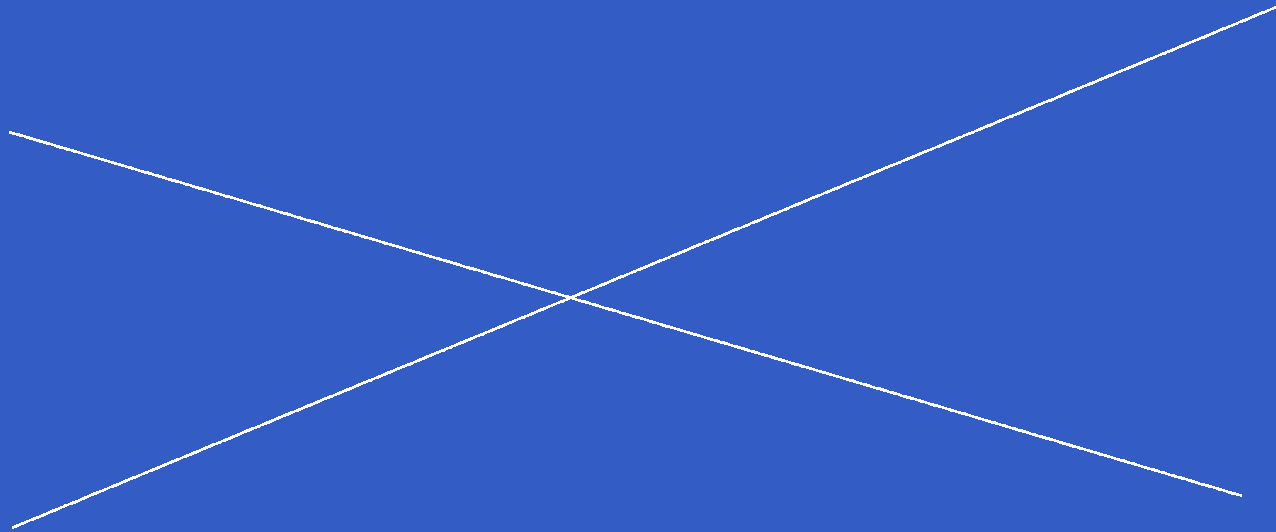
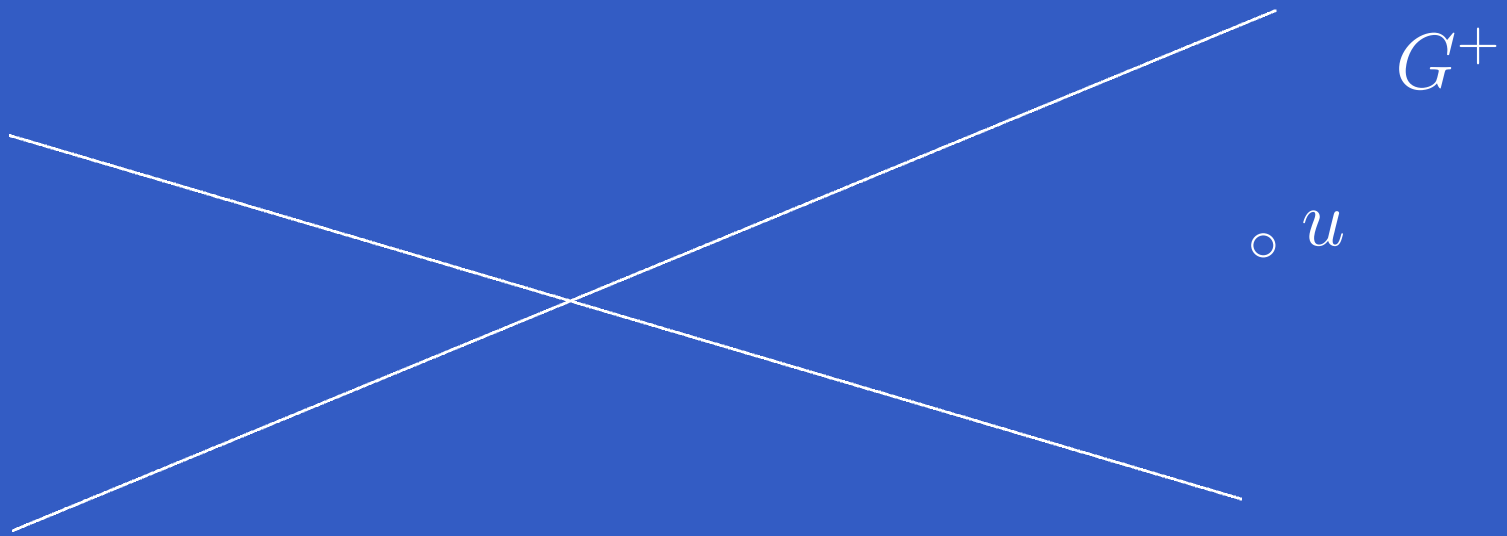
The functor Γ *defines a categorical equivalence between the category of PMV-algebras and the category of unital* ℓ*-groups.*
- $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ - PMV-algebra such that $x^{\sim} = x^{-}$ (symmetric) but not necessarily MV-algebra



G^+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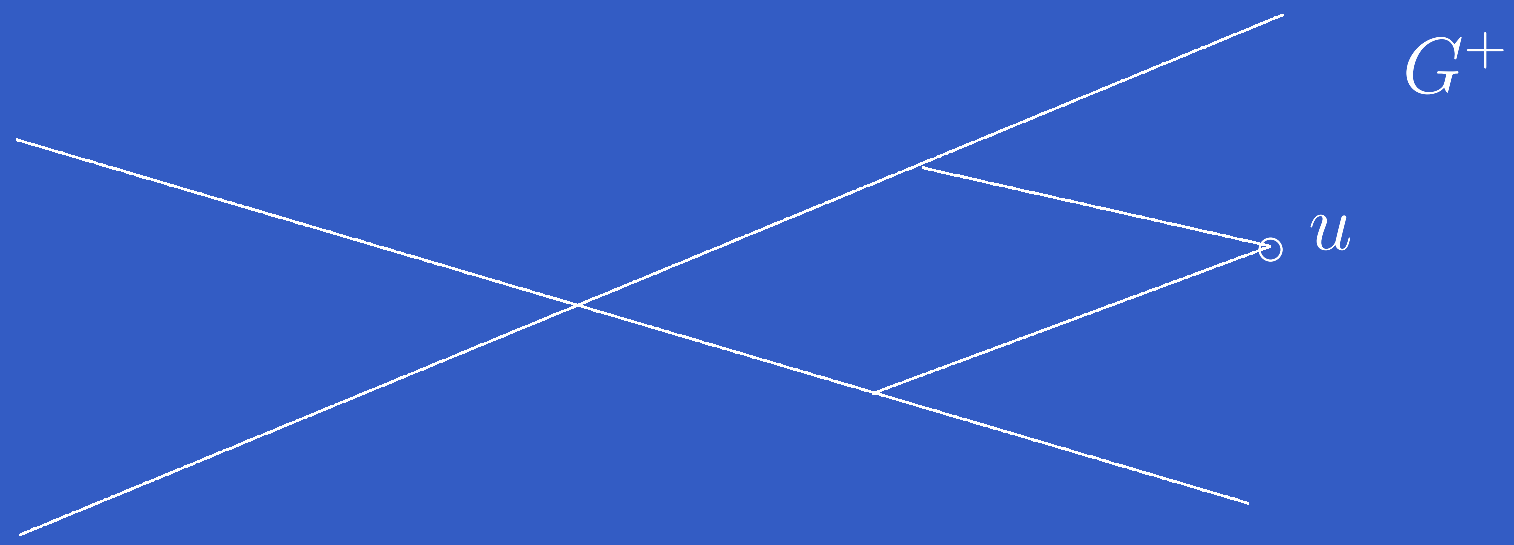
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- Let u be the translation $u(t) = t + 1$, $t \in \mathbb{R}$,

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Pseudo EMV-algebras

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- $\forall\, x \in M$, $\exists\, a \in \mathcal{I}(M)$ s.t. $x \leq a$.

$$\lambda_a(x) := \min\{z \in [0, a] \colon\ z \oplus x = a\},\ a \in \mathcal{I}(M)$$
$$\rho_a(x) := \min\{z \in [0, a] \colon\ x \oplus z = a\}$$

and $([0, a]; \oplus, \lambda_a, \rho_a, 0, a)$ is a PMV-algebra

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- Many properties if commutative EMV-algebras hold also for PEMV-algebras
- Every PEMV-algebra with top element is equivalent to PMV-algebra
- Every PEMV-algebra M either has top element or it can be embedded into a PEMV N with 1 as a maximal and normal ideal of N, and every element is either in the image of M or is a complement of some element from the image of M.

- Not a variety

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- M has a maximal ideal but not necessarily normal

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- integral representation of (finitely additive) states as integral through σ-additive probability measure.

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- $x \ominus y = (x - y) \vee 0$, $x \odot\!\sim y = (-x + y) \vee 0$

Representation of wPEMV-algebras

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- How it is in the case of PEMV-algebras?
- Basic representation theorem holds for commutative, representable wPEMV-algebras, for other it is an open question.

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- partial operation $+$ on M: $x+y$ is defined iff $(x \oplus y) \ominus y = x$, and we define $x+y = x \oplus y$
- $(M;+,0)$ is a GPEA with RDP$_2$:
- if $a_1 + a_2 = b_1 + b_2$, $\exists\, c_{11}, c_{12}, c_{21}, c_{22} \in M$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$, and $c_{12} \wedge c_{21} = 0$.

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- **Theorem 0.12** *A wPEMV-algebra without top element can be represented as a maximal and normal ideal of N with top element iff M possesses a left unitizing automorphism.*

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- Does every wPEMV-algebra without top element posses a left unitizing automorphism?

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- $\mathcal{C}an$ $x = (x \oplus y) \ominus y$, $x = y \odot \sim (y \oplus x)$
- $\mathcal{C}an$ has uncountably many subvarieties
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Varieties of wPEMV-algebras

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- Wcom weakly commutative is a variety
- Repr is a variety

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- normal valued M every value of x is normal in its cover
- NV is a variety containing Repr
- Also for this kind of algebras, the research of Charles Holland was influenced

Dedication

- The talk is dedicated to memory of Charles Holland

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- Thanks for your interest!