Twist structures and Nelson conuclei

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Residuated lattices

A *residuated lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, e)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, e)$ is a monoid and
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Examples/connections/applications:

1. Ideals of rings, under the usual multiplication and division of ideals. (Ward and Dilworth)
2. Lattice-ordered groups: $x\backslash y = x^{-1}y,\ y/x = yx^{-1}$.
3. Boolean and Heyting algebras, where $x \cdot y = x \wedge y$ (add a constant 0).
4. Relation algebras: $R\backslash S = (R^{\cup} \circ S^c)^c, \quad S/R = (S^c \circ R^{\cup})^c$.
5. Mathematical linguistics: Context-free grammars, pregroups. (Lambek)
6. CS: Memory allocation, pointer management, concurrent programming. (Separation logic, bunched implication logic).
7. Substructural logics: Linear, relevance, MV, BL, MTL, where multiplication is strong conjunction. Connections to proof theory.

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A *cyclic involutive residuated lattice* is an expansion of a residuated lattice with a unary *negation* operation $\sim$ such that

$$\sim\sim x = x \text{ (double negation) and } x\backslash y = \sim x/\sim y \text{ (contraposition).}$$

(In the commutative case contraposition reads: $x \to y = \sim y \to \sim x$.)
We define the *negation constant* $f = \sim e$; then f is *cyclic*: $x\backslash f = f/x$, for all x. This gives an alternative term equivalent definition via $\sim x = x\backslash f$.

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A *residuated lattice-ordered semigroup* is defined as above by without asking for an identity element.

Kalman residuated lattices

A *Kalman residuated lattice* is a odd involutive commutative residuated lattice $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, e)$ satisfying the following equations:

1. $(x \cdot y) \wedge e = (x \wedge e) \cdot (y \wedge e)$,
2. $e \wedge (x \vee y) = (e \wedge x) \vee (e \wedge y)$ and $x \wedge (y \vee e) = (x \wedge y) \vee (x \wedge e)$,
3. $((x \wedge e) \to y) \wedge (x \to (y \vee e)) = x \to y$ [equivalently, $((x \wedge e) \to y) \wedge ((\sim y \wedge e) \to \sim x) = x \to y$].

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A *Nelson residuated lattice* is a integral involutive commutative residuated lattice $\mathbf{A} = (A, \vee, \wedge, \cdot, \to, e)$, satisfying the following condition

1. $(x^2 \to y) \wedge ((\neg y)^2 \to \neg x) = x \to y$

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We will unify and generalize these classes of algebras.

Hidden conuclei

In each of these cases, we consider the (term-definable) map $\mathbf{n}$:

- Kalman (and NPc) residuated lattices: $\mathbf{n}(x) = x \wedge e$.
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A *weak conucleus* δ on a residuated lattice-ordered semigroup $\mathbf{A} = (A, \vee, \wedge, \cdot, \backslash, /)$ is an interior operator ($\delta(x) \leq x$, $\delta(\delta(x)) = \delta(x)$, if $x \leq y$ then $\delta(x) \leq \delta(y)$) that satisfies:

$$\delta(x) \cdot \delta(y) \leq \delta(x \cdot y).$$

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If $\mathbf{A}$ is a residuated lattice with neutral element e and δ additionally satisfies: $\delta(e) \cdot \delta(x) = \delta(x) \cdot \delta(e) = \delta(x)$ then δ is called a *conucleus* on $\mathbf{A}$.

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Given a residuated lattice $\mathbf{A}$ and a conucleus δ on $\mathbf{A}$ the algebra

$$\mathbf{A}_\delta = \delta[\mathbf{A}] = (\delta[A], \vee, \wedge_\delta, \cdot, \backslash_\delta, /_\delta, \delta(e))$$

where $x \wedge_\delta y = \delta(x \wedge y)$, $x \backslash_\delta y = \delta(x \backslash y)$ and $y /_\delta x = \delta(y/x)$ for $x, y \in \delta[A]$, is also a residuated lattice. (We use weak conuclei on a residuated lattice-ordered semigroups.)

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Fact. In the above examples, the map $\mathbf{n}$ is a conucleus. Does it satisfy any special properites?

Nelson conucleus

Given residuated lattice $\mathbf{A}$, a *Nelson conucleus* is an interior operator $\mathbf{n}$ such that:

1. $\mathbf{n}(x \vee y) = \mathbf{n}(x) \vee \mathbf{n}(y)$
2. $\mathbf{n}(xy) = \mathbf{n}(x)\mathbf{n}(y)$
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In the involutive case (3) can be rephrased in terms of the divisions and involution:

Lemma: Let $\mathbf{A}$ be an involutive residuated lattice and $\mathbf{n}$ a conucleus on $\mathbf{A}$. The operator $\mathbf{n}$ satisfies (3) iff it satisfies one of the following equivalent identities

5. $x\backslash y = (\mathbf{n}(x)\backslash y) \wedge ({\sim}x/\mathbf{n}({\sim}y))$ [equivalently, $x\backslash y = (\mathbf{n}(x)\backslash y) \wedge (x\backslash{\sim}\mathbf{n}({\sim}y))$]
   $y/x = (y/\mathbf{n}(x)) \wedge (\mathbf{n}({\sim}y)\backslash{\sim}x)$ [equivalently, $y/x = (y/\mathbf{n}(x)) \wedge ({\sim}\mathbf{n}({\sim}y)/x)$]

In this case we also have that $\mathbf{n}({\sim}e)$ is cyclic in $\mathbf{A}_{\mathbf{n}}$.

Let $\mathcal{NCA}$ be the variety of *Nelson conucleus algebras* $(\mathbf{A}, \mathbf{n})$ where $\mathbf{A}$ is a cyclic involutive residuated lattice and $\mathbf{n}$ is a Nelson conucleus on $\mathbf{A}$.

Note that Kalman lattices, Nelson residuated lattices and NPc-residuated lattices are contained in $\mathcal{NCA}$.

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In this case we also have that $\mathbf{n}({\sim}e)$ is cyclic in $\mathbf{A_n}$.

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Can we link the residuated lattice $\mathbf{A_n}$ (and $\mathbf{n}({\sim}e)$) back to $\mathbf{A}$?

Twist products

Given a residuated lattice-ordered semigroup $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /)$, we consider the set $Tw(L) = L \times L$ and define the operations $\wedge$ and $\vee$ as in $\mathbf{L} \times \mathbf{L}^\partial$. Also, for $a, a', b, b' \in L$, we define:

$$\sim(a, b) = (b, a)$$

$$(a, b) \cdot (a', b') = (a \cdot a', b'/a \wedge a'\backslash b)$$

$$(a, b)\backslash(a', b') = (a\backslash a' \wedge b/b', b' \cdot a)$$

$$(a', b')/(a, b) = (a'/a \wedge b'\backslash b, a \cdot b')$$

The resulting structure $\mathbf{Tw}(\mathbf{L}) = (\mathsf{Tw}(L), \wedge, \vee, \cdot, \backslash, /, \sim)$ is an involutive residuated lattice-ordered semigroup, and is called the *full twist structure/product* over $\mathbf{L}$.

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If $\mathbf{L}$ is a *topped residuated lattice* with neutral element e and top element $\top$ the structure $(\mathsf{Tw}(L), \wedge, \vee, \cdot, \backslash, /, \sim, (e, \top))$ is an involutive residuated lattice [Tsinakis-Wille]; in that case we will add the multiplicative identity element to the signature.

Twist products were introduced by Kahlman for lattices and Chu for categories. They were used by Tsinakis-Wille to construct atomic varieties. They feature in the study of bilattices. They are used in constructing Sugihara monoids from Gödel algebras.

Double-division conucleus

Let $\mathbf{A} = (A, \vee, \wedge, \cdot, \backslash, /)$ be a residuated lattice-ordered semigroup and $p \in A$ an idempotent element ($p = p^2$) that is also *positive* ($p\backslash x, x/p \leq x \leq px, xp$, for all x). Note that if $\mathbf{A}$ has an identity element e, then p is positive iff $e \leq p$.

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It is shown in [G.-Jipsen] that the map defined by

$$\delta_p(x) = p\backslash x/p$$

is a weak conucleus on $\mathbf{A}$, that p is an identity element for $\delta_p[A] = p\backslash A/p = \{p\backslash a/p : a \in A\}$ and that $\delta_p[A] = \{a \in A : ap = a, pa = a\}$.
Therefore the *localized* algebra

$$p\backslash \mathbf{A}/p = \delta_p[\mathbf{A}] = (p\backslash A/p, \wedge, \vee, \cdot, \backslash, /, p)$$

forms a residuated lattice, called the *double-division conucleus image of* $\mathbf{A}$ *by* p, and $p\backslash \mathbf{A}/p$ is a subalgebra of $\mathbf{A}$.

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Also, if $\mathbf{A}$ is (cyclic) involutive with involution $\sim$, then $p\backslash \mathbf{A}/p$ is also (cyclic) involutive and $p\backslash \mathbf{A}/p$ is a subalgebra of $\mathbf{A}$ with respect to the operations $\wedge, \vee, \cdot, \backslash, /, \sim$.

This construction plays an important role in GBI-algebras and in weakening relation algebras.

Localized twist products

Lemma: Let $\mathbf{L}$ be a residuated lattice and $\imath \in L$.

1. The element $(e, \imath)$ is a positive idempotent of the residuated lattice-ordered semigroup $\mathbf{Tw}(\mathbf{L})$.
2. The element (a, b) is fixed by $\delta_{(e,\imath)}$ iff $ab \vee ba \leq \imath$. (Recall: $\delta_p(x) = p\backslash x/p$.)

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Given a residuated lattice $\mathbf{L}$ and $\imath \in L$, we define the following *twist structure* over $(\mathbf{L}, \imath)$:

$$\mathbf{Tw}(\mathbf{L}, \imath) = \delta_{(e,\imath)}[\mathbf{Tw}(\mathbf{L})] = (e, \imath)\backslash\mathbf{Tw}(\mathbf{L})/(e, \imath)$$

Even though $\mathbf{Tw}(\mathbf{L})$ may lack an identity element (when $\mathbf{L}$ lacks a top), $\mathbf{Tw}(\mathbf{L}, \imath)$ is still a residuated lattice with identity element $(e, \imath)$.

Localized twist products

We have shown that $\mathsf{Tw}(\mathbf{L}, \imath) = \{(a, b) \in L \times L^\partial : ab \vee ba \leq \imath\}$

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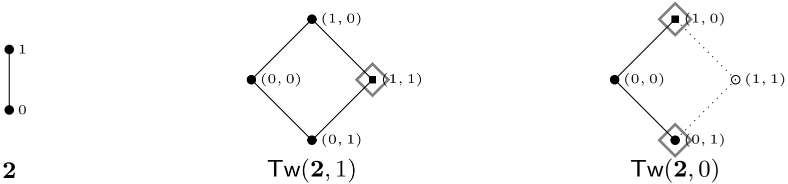
Lemma: $\mathsf{Tw}(\mathbf{L}, \imath)$ is a downset of the direct product lattice $\mathbf{L} \times \mathbf{L}$. Moreover, if $\imath$ is cyclic, then $M_\imath = \{(a, b) : a = \imath/b \wedge b\backslash\imath \text{ and } b = \imath/a \wedge a\backslash\imath\}$ consists of the maximal elements of $\mathsf{Tw}(L, \imath)$ and $\mathsf{Tw}(L, \imath) = \downarrow_{\mathbf{L}\times\mathbf{L}} M_\imath$.



The two twist structures on the 2-element residuated lattice $\mathbf{2}$. The identity $(1, \imath)$ is a bold square and the maximal sets are within light squares.

Known representations

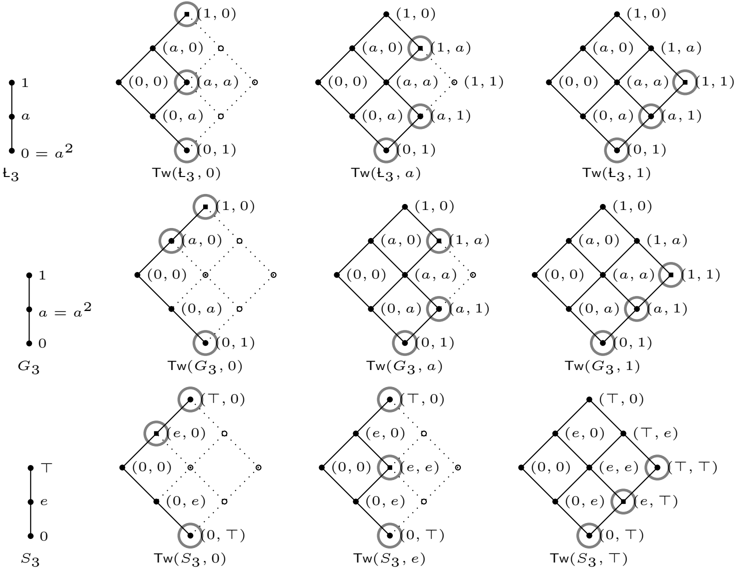
Theorem. (Busaniche-Cignoli, 2014) If $\mathbf{L} = (L, \vee, \wedge, \cdot, \to, 1)$ is an integral commutative residuated lattice, then $\mathbf{Tw}(\mathbf{L}, 1)$ is a Kalman lattice. Moreover, for each Kalman lattice $\mathbf{A}$ there is an integral residuated lattice $\mathbf{L}$ such that $\mathbf{A}$ is a subalgebra of $\mathbf{Tw}(\mathbf{L}, 1)$.

Theorem. (Busaniche-Cignoli, 2010) If $\mathbf{H} = (H, \vee, \wedge, \to, 0, 1)$ is a Heyting algebra then $\mathbf{Tw}(\mathbf{H}, 0)$ is a Nelson residuated lattice with bottom element $(0, 1)$ (added to the signature). Moreover, every Nelson residuated lattice is embeddable in $\mathbf{Tw}(\mathbf{H}, 0)$

Theorem. (Aguzzoli-Busaniche-Gerla-Marcos, 2017) If $\mathbf{H} = (H, \vee, \wedge, \to, 1)$ is a Brouwerian algebra, then $\mathbf{Tw}(\mathbf{H}, 1)$ is a Nelson paraconsistent residuated lattice. Also, every NPc-lattice $\mathbf{A}$ can be embedded in $\mathbf{Tw}(\mathbf{H}, 1)$ for some Heyting algebra $\mathbf{H}$.

We will unify and generalize these representations. In particular we need a way to find $\mathbf{A}$ inside the twist structure $\mathbf{Tw}(\mathbf{L}, \imath)$. This is done by a Nelson conucleus $\mathbf{n}$ on $\mathbf{Tw}(\mathbf{L}, \imath)$, whose image is $\mathbf{A}$.

Pictures


1
a
$0 = a^2$
Ł3
(1, 0)
(a, 0)
(0, 0)
(a, a)
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Tw(G_3, 1)
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e
0
S_3
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(e, 0)
(0, 0)
(0, e)
(0, ⊤)
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(⊤, 0)
(e, 0)
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Tw(S_3, e)
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Categorical adjunction

We consider the category $\mathcal{RL}_{\mathsf{cy}}$ with objects algebras $(\mathbf{L}, \imath)$, where $\mathbf{L}$ is a residuated lattice and $\imath$ is a cyclic element of $\mathbf{L}$; the morphisms are homomorphisms of these algebras (they preserve the cyclic element). Also, note that $\mathcal{NCA}$ defines a category with objects algebras $(\mathbf{A}, \mathbf{n})$ such that $\mathbf{A}$ is a (cyclic) involutive residuated lattice and $\mathbf{n}$ is a Nelson conucleus on $\mathbf{A}$; the morphisms are the algebraic homomorphisms.

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$$\mathbf{T}(\mathbf{L}, \imath) = (\mathbf{Tw}(L, \imath), \mathbf{n}_{\mathsf{Tw}}) \quad \text{and} \quad \mathbf{T}(f)(a, b) = (f(a), f(b)).$$

It can be easily verified that $\mathbf{T}$ is a functor from $\mathcal{RL}_{\mathsf{cy}}$ to $\mathcal{NCA}$.

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Also, for and object $(\mathbf{A}, \mathbf{n}) \in \mathcal{NCA}$ and a homomorphism $f : (\mathbf{A}, \mathbf{n}) \to (\mathbf{B}, \mathbf{n}')$ in $\mathcal{NCA}$ we define

$$\mathbf{R}(\mathbf{A}, \mathbf{n}) = (\mathbf{A}_{\mathbf{n}}, \mathbf{n}(\sim e)) \quad \text{and} \quad \mathbf{R}(f) \text{ to be the restriction of } f \text{ to } \mathbf{A}_{\mathbf{n}}.$$

It can be shown that $\mathbf{R}$ is a functor from $\mathcal{NCA}$ to $\mathcal{RL}_{\mathsf{cy}}$.

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Theorem The functors $\mathbf{R}$ and $\mathbf{T}$ form an adjunction. The unit and counit are:

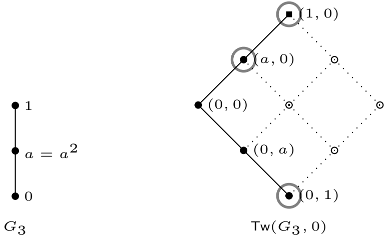
$$\psi_{(\mathbf{L},\imath)} : (\mathbf{L}, \imath) \to \mathbf{RT}(\mathbf{L}, \imath) \text{ given by } \psi_{(\mathbf{L},\imath)}(a) = (a, a \backslash \imath)$$

and

$$\phi_{(\mathbf{A},\mathbf{n})} : (\mathbf{A}, \mathbf{n}) \to \mathbf{TR}(\mathbf{A}, \mathbf{n}) \text{ given by } \phi_{(\mathbf{A},\mathbf{n})}(x) = (\mathbf{n}(x), \mathbf{n}(\sim x)).$$

Not a categorical equivalence

The functors $\mathbf{R}$ and $\mathbf{T}$ do not form an equivalence between the categories $\mathcal{RL}_{\mathsf{cy}}$ and $\mathcal{NCA}$, as $\phi_{(\mathbf{A},\mathbf{n})}$ is not always an isomorphism. For example, $S = \mathsf{Tw}(\mathbf{G}_3, 0) \setminus \{(0,0)\}$ is the universe of a subalgebra $\mathbf{S}$ of $\mathbf{Tw}(\mathbf{G}_3, 0)$, such that $\mathbf{n}_{\mathsf{Tw}}(S) = G_3$.



Then the function

$$\phi_{(\mathbf{S},\mathbf{n}_{\mathsf{Tw}})} : (\mathbf{S}, \mathbf{n}_{\mathsf{Tw}}) \to \mathbf{TR}(\mathbf{S}, \mathbf{n}_{\mathsf{Tw}})$$

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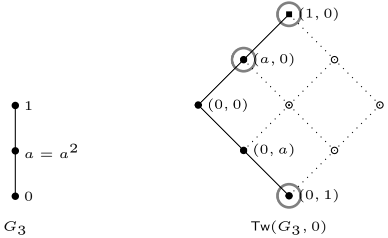

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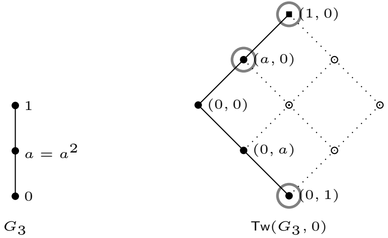

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Thank you!

Nelson-type algebras and categorical equivalence

The subvariety $\mathcal{NT}$ (*Nelson-type algebras*) of $\mathcal{NCA}$ consists of $(\mathbf{A}, \mathbf{n})$, where $\mathbf{A}$ is a commutative, distributive, involutive residuated lattice and $\mathbf{n}(x) = (x \wedge e)^2$.

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If $\mathbf{H}$ is a Brouwerian algebra, a filter F of $\mathbf{H}$ is called a *Boolean filter* if it contains all *dense* elements $x \vee (x \to y)$, for $x, y \in H$. Then $\mathbf{H}/F$ is a generalized Boolean algebra.

Lemma: Let $\mathbf{H}$ be a Brouwerian algebra, $\imath \in H$ and F a Boolean filter of $\mathbf{H}$. Then

$$\mathsf{Tw}(H, \imath, F) = \{(a, b) \in H \times H^\partial : a \wedge b \leq \imath, a \vee b \in F\}$$

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Theorem: The category $\mathcal{NT}$ is equivalent to the category with objects triples $(\mathbf{H}, \imath, F)$, where $\mathbf{H}$ is a Brouwerian algebra, $\imath \in H$ and F is a Boolean filter of $\mathbf{H}$ and the morphisms are Brouwerian algebra homomorphisms $f : \mathbf{H}_1 \to \mathbf{H}_2$ such that $f(\imath_1) = \imath_2$ and $f(F_1) \subseteq F_2$.

Involutive case: categorical equivalence

Lemma: If $\mathbf{L}$ is an involutive commutative residuated lattice, $\imath \in L$ and F a filter of L with $e + \imath = f \to \imath \in F$, then $\mathsf{Tw}(L, \imath, F)$ is a subalgebra of $\mathbf{Tw}(\mathbf{L}, \imath)$, where

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So, for all $x \in A_n$, we have $\neg_{\mathbf{n}} \neg_{\mathbf{n}} x = x$ and so $\mathbf{A}_n$ is an integral commutative involutive residuated lattice (the bottom element 0 is added to the type).

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1. $F_A := \{\mathbf{n}(x) + \mathbf{n}(\sim x) : x \in A\} = \{\mathbf{n}(\sim z) : \mathbf{n}(z) = 0, z \in A\}$.
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Theorem: This extends to a categorical equivalence.

Rasiowa-style presentation

If $(\mathbf{A}, \mathbf{n}) \in \mathcal{NCA}$, then unfortunately we do not have $\mathbf{n}(x \backslash y) = x \backslash_{\mathbf{n}} y$. However, we have $\mathbf{n}(x \backslash\!\backslash y) = x \backslash_{\mathbf{n}} y$, where

$$x \backslash\!\backslash y = \mathbf{n}(x) \backslash y \quad \text{and} \quad y /\!/ x = y / \mathbf{n}(x).$$

Lemma: For a residuated lattice $\mathbf{A}$ and a Nelson conucleus $\mathbf{n}$ on $\mathbf{A}$ we have that the map $n : A \to A_{\mathbf{n}}$, (where $n(x) = \mathbf{n}(x)$), is a homomorphism from $\bar{\mathbf{A}} = (A, \wedge, \vee, \cdot, \backslash\!\backslash, /\!/, e)$ into $\mathbf{A}_{\mathbf{n}} = (\mathbf{n}[\mathbf{A}], \wedge_{\mathbf{n}}, \vee, \cdot, \backslash_{\mathbf{n}}, /_{\mathbf{n}}, \mathbf{n}(e))$.

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We define the relations θ and $\preceq$ on A by:

$$x\theta y \text{ iff } \mathbf{n}(x) = \mathbf{n}(y)$$
$$x \preceq y \text{ iff } \mathbf{n}(x) \leq \mathbf{n}(y)$$

Lemma: If $\mathbf{n}$ is a Nelson conucleus on a residuated lattice $\mathbf{A}$, then:

1. The relation $\preceq$ is a preorder compatible with the operations of $\bar{\mathbf{A}}$. Also, $x\theta y$ iff $x \preceq y$ and $y \preceq x$.
2. The relation θ is a congruence on $\bar{\mathbf{A}}$, which is the kernel of the map n. Thus, $\bar{\mathbf{A}}/\theta$ is isomorphic to $\mathbf{A_n}$.

Rasiowa-style presentation

A *Rasiowa-type algebra* is an algebra $\bar{\mathbf{A}} = (A, \vee, \wedge, \cdot, \backslash\!\!\backslash, /\!\!/, \sim, e)$ where

1. $(A, \vee, \wedge)$ is a lattice, $\sim\sim x = x$ and $\sim(x \vee y) = \sim x \wedge \sim y$;
2. the relation $\preceq$ is a preorder, where $x \preceq y$ if and only if $(x \backslash\!\!\backslash y) \backslash\!\!\backslash (x \backslash\!\!\backslash y) \leq (x \backslash\!\!\backslash y)$, and also if and only if $(y /\!\!/ x) /\!\!/ (y /\!\!/ x) \leq (y /\!\!/ x)$;
3. the equivalence relation θ induced by $\preceq$ is a congruence on $\bar{\mathbf{A}} = (A, \vee, \wedge, \cdot, \backslash\!\!\backslash, /\!\!/, e)$ and the quotient algebra $\bar{\mathbf{A}}/\theta$ is a residuated lattice;
4. $\sim(x \backslash\!\!\backslash y)\ \theta\ (\sim y \cdot x)$, $\sim(y /\!\!/ x)\ \theta\ (x \cdot \sim y)$ and $\sim(x \cdot y)\ \theta\ (y \backslash\!\!\backslash \sim x) \wedge (\sim y /\!\!/ x)$;
5. $x \leq y$ if and only if $x \preceq y$ and $\sim y \preceq \sim x$.
6. For each $x \in A$, $x \backslash\!\!\backslash \sim e = \sim e /\!\!/ x$.

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4. $\sim(x \mathbin{\backslash\!\backslash} y) \; \theta \; (\sim y \cdot x)$, $\sim(y \mathbin{/\!/} x) \; \theta \; (x \cdot \sim y)$ and $\sim(x \cdot y) \; \theta \; (y \mathbin{\backslash\!\backslash} \sim x) \wedge (\sim y \mathbin{/\!/} x)$;
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Theorem: Rasiowa-type algebras are term equivalent to $\mathcal{NCA}$'s, via

$$x \mathbin{\backslash\!\backslash} y = \mathbf{n}(x) \backslash y, \quad y \mathbin{/\!/} x = y / \mathbf{n}(x)$$

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Nelson lattices and paraconsistent Nelson lattices were defined by Rasiowa as special cases of Rasiowa-type algebras. In our presentation, the elements of $\mathbf{A_n}$ serve as canonical representatives for the elements of $\bar{\mathbf{A}}/\theta$ and the description is internal to $\mathbf{A}$.