Hausdorff reflection of internal preneighbourhood spaces

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Topological Spaces

Definition (Topological Spaces, see Schubert, *Topology*, §2.3)

A *topological space* is given by (X, η), where X is a set and $X \xrightarrow{\eta} \mathtt{Fil}X$ is a function assigning to each $x \in X$ a filter η_x (called the *neighbourhood filter* of x) such that:

$$U \in \eta_x \Rightarrow x \in U, \quad \text{and}$$

$$U \in \eta_x \Rightarrow (\exists V \in \eta_x)\big(y \in V \Rightarrow U \in \eta_y\big).$$

Spaces internalised

The notion of a *space* is now seen inside a large number of categories, see Ghosh, "Internal neighbourhood structures".

Spaces internalised

A *context* is a category $\mathbb{A}$ with the following properties:

(a) $\mathbb{A}$ is finitely complete

see Ghosh, “Internal neighbourhood structures”, §2, Ghosh, “Internal neighbourhood structures II: Closure and Closed Morphisms”, §1

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(c) $\mathbb{A}$ has a proper (E, M)-factorisation structure

(d) For each object X of $\mathbb{A}$, the (possibly large) set $\mathtt{Sub}_{\mathsf{M}}(X)$ of admissible subobjects of X is a complete lattice.

see Ghosh, "Internal neighbourhood structures", §2, Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", §1

Spaces internalised

Example (Contexts abound. . . , see Ghosh, "Internal neighbourhood structures", Examples in §3)

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(b) $(\mathsf{Set}, \mathsf{Surjection}, \mathsf{Injection})$

(c) $(\mathtt{Top}, \mathtt{Epi}, \mathtt{ExtMon})$

(d) $(\mathtt{Meas}, \mathtt{Epi}, \mathtt{ExtMon})$

(e) $(\mathtt{Grp}, \mathtt{RegEpi}, \mathtt{Mon})$

(f) $((\Omega, \Xi)\text{-}\mathtt{Alg}, \mathtt{RegEpi}, \mathtt{Mon})$

(g) $(\mathtt{Loc}, \mathtt{Epi}, \mathtt{RegMon})$

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(l) $(\mathbb{A}, \mathtt{Epi}(\mathbb{A}), \mathtt{ExtMon}(\mathbb{A}))$, where $\mathbb{A}$ is a small complete and small cocomplete well powered category

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(k) if $(\mathbb{A}, \mathsf{E}, \mathsf{M})$ be a context then for any object B, then $((\mathbb{A} \downarrow B), (\mathsf{E} \downarrow B), (\mathsf{M} \downarrow B))$ is also a context
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Spaces internalised

Definition (Filters)

Given any object X, a *filter* F on X is a subset of $\mathtt{Sub}_{\mathrm{M}}(X)$ such that

(a) $x \geq y \in F \Rightarrow x \in F$, and

(b) $x, y \in F \Rightarrow x \wedge y \in F$

The set of all filters on X is $\mathtt{Fil}X$.

Spaces internalised

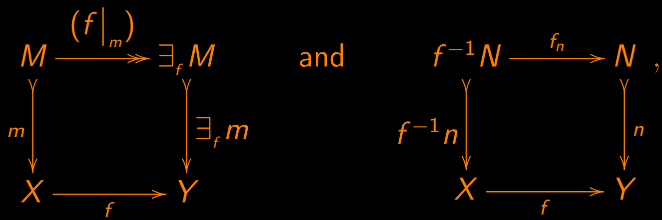
$\mathtt{Fil}X$ is a complete algebraic lattice, with compact elements being

$$\uparrow x = \{p \in \mathtt{Sub}_{\mathsf{M}}(X) : x \leq p\}.$$

$\mathtt{Fil}X$ is distributive if and only if $\mathtt{Sub}_{\mathsf{M}}(X)$ is distributive (see Iberkleid and McGovern, "A natural equivalence for the category of coherent frames", Theorem 1.2).

Spaces internalised

The (E, M) factorisation produces the notion of *image* and *preimage* of a morphism $X \xrightarrow{f} Y$:



(E, M) factorisation of $f \circ m$			pullback of n along f
$\exists_f m$	image of m under f	$f^{-1}n$	preimage of n under f
$\exists_f M$	image of M under f	$f^{-1}N$	preimage of N under f
$(f\|_m)$	restriction of f to m	f_n	corestriction of f to n

Spaces internalised

The image and preimage assignments for a morphism $X \xrightarrow{f} Y$ constitute an adjunction

$$\mathtt{Sub}_{\mathsf{M}}(X) \underset{f^{-1}}{\overset{\exists_f}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathtt{Sub}_{\mathsf{M}}(Y)\ .$$

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Definition (Neighbourhoods, see Ghosh, "Internal neighbourhood structures", Definition 3.1)

Let X be an object of $\mathbb{A}$.

Spaces internalised

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Let X be an object of $\mathbb{A}$.

(a) A *preneighbourhood system* on X is an order preserving function $\mathtt{Sub}_{\mathsf{M}}(X)^{\mathrm{op}} \xrightarrow{\mu} \mathtt{Fil}X$ such that for each $x \in \mathtt{Sub}_{\mathsf{M}}(X)$

$$p \in \mu(x) \Rightarrow x \leq p.$$

The pair (X, μ) is called an *internal preneighbourhood space* of $\mathbb{A}$.

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(b) A preneighbourhood system μ on X is a *weak neighbourhood system* if

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(c) A weak neighbourhood system μ on X is a *neighbourhood system* on X if

$$\mu\Big(\bigvee_{i \in I} p_i\Big) = \bigcap_{i \in I} \mu(p_i).$$

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Spaces internalised

Definition (Morphisms of Neighbourhoods, see Ghosh, "Internal neighbourhood structures", Definition 3.39)

Let (X, μ), (Y, ϕ) be internal preneighbourhood spaces of $\mathbb{A}$ and $X \xrightarrow{f} Y$ be a morphism of $\mathbb{A}$.

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Let (X, μ), (Y, ϕ) be internal preneighbourhood spaces of $\mathbb{A}$ and $X \xrightarrow{f} Y$ be a morphism of $\mathbb{A}$.

(a) The morphism f is a *preneighbourhood morphism*, written $(X, \mu) \xrightarrow{f} (Y, \phi)$, if for each $y \in \mathtt{Sub}_{\mathsf{M}}(Y)$

$$p \in \phi(y) \Rightarrow f^{-1}p \in \mu(f^{-1}y).$$

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(b) If (X, μ) and (Y, ϕ) are internal neighbourhoods of $\mathbb{A}$ then a preneighbourhood morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ is a *neighbourhood morphism* if for any family $\langle y_i : i \in I \rangle$ of admissible subobjects of Y

$$f^{-1}(\bigvee_{i \in I} y_i) = \bigvee_{i \in I} f^{-1} y_i.$$

Spaces internalised

Definition (Categories of Neighbourhoods, see Ghosh, "Internal neighbourhood structures", Definition 4.1)

(a) $\mathtt{pNbd}[\mathbb{A}]$ is the category of all internal preneighbourhood spaces of $\mathbb{A}$ and preneighbourhood morphisms.

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(c) $\mathtt{Nbd}[\mathbb{A}]$ is the category of all internal neighbourhood spaces of $\mathbb{A}$ and neighbourhood morphisms.

Spaces internalised

Theorem (Topologicity, see Ghosh, "Internal neighbourhood structures", Theorem 4.8)

The categories $\mathtt{pNbd}[\mathbb{A}]$ *and* $\mathtt{wNbd}[\mathbb{A}]$ *are topological over* $\mathbb{A}$*.*

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The category $\mathtt{Nbd}[\mathbb{A}]$ *is topological over* $\mathbb{A}$ *provided preimage for every morphism preserve joins.*

Closure et al...

Definition (Definition of closure, closed subobject and closed morphism, see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", definitions in §3)

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Let (X, μ) be an internal preneighbourhood space and $p \in \mathtt{Sub}_{\mathsf{M}}(X)$. The admissible subobject:

$$\mathrm{cl}_{\mu} p = \bigvee \left\{ u \in \mathtt{Sub}_{\mathsf{M}}(X)_{\neq 1} : x \in \mu(u) \Rightarrow x \wedge p \neq \sigma_X \right\} \tag{1}$$

is called the *μ-closure of p*.

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The subobject p is *μ-closed* if $\mathrm{cl}_\mu p = p$.

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For any internal preneighbourhood space (X, μ), $\mathfrak{C}_\mu = \left\{ p \in \mathtt{Sub}_{\mathsf{M}}(X) : p = \mathrm{cl}_\mu p \right\}$.

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Given the internal preneighbourhood spaces (X, μ) and (Y, ϕ), a morphism $X \xrightarrow{f} Y$ is said to be *μ-ϕ closed* or simply *closed* if it *preserves* closed subobjects, i.e., $p \in \mathfrak{C}_{\mu} \Rightarrow \exists_f p \in \mathfrak{C}_{\phi}$.

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$\mathbb{A}_{\mathsf{cl}}$ is the (possibly large) set of all closed morphisms of $\mathbb{A}$.

Properties of the closure operator

Definition (Closure Operation)

An order preserving function $P \xrightarrow{c} P$ on a partially ordered set P is said to be a *closure operation* if it satisfies the conditions

$$x \leq c(x) \text{ (Extensionality)}$$

and

$$c \circ c = c \text{ (Idempotence)}$$

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and

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If it further satisfies

$$\mathsf{c}(x \vee y) = \mathsf{c}(x) \vee \mathsf{c}(y) \text{ (Additivity)}$$

then it is called a *Kuratowksi closure operation*.

Properties of the closure operator

Theorem (Properties of Closure, see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Theorem 3.1)

- *Given any internal preneighbourhood space* (X, μ)*, the function* $\mathtt{Sub}_{\mathsf{M}}(X) \xrightarrow{\mathrm{cl}_\mu} \mathtt{Sub}_{\mathsf{M}}(X)$ *defines a closure operation on* $\mathtt{Sub}_{\mathsf{M}}(X)$ *such that* $\mathrm{cl}_\mu \sigma_X = \sigma_Y$*.*

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Theorem (Properties of Closure, see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Theorem 3.1)

- *Given any internal preneighbourhood space* (X, μ), *the function* $\mathtt{Sub}_{\mathsf{M}}(X) \xrightarrow{\mathrm{cl}_\mu} \mathtt{Sub}_{\mathsf{M}}(X)$ *defines a closure operation on* $\mathtt{Sub}_{\mathsf{M}}(X)$ *such that* $\mathrm{cl}_\mu \sigma_X = \sigma_Y$.
- *If* $(X, \mu) \xrightarrow{f} (Y, \phi)$ *is a preneighbourhood morphism reflecting zero then it is* μ-ϕ continuous, *i.e., for any* $p \in \mathtt{Sub}_{\mathsf{M}}(X)$:

$$\exists_f \mathrm{cl}_\mu p \leq \mathrm{cl}_\phi \exists_f p \qquad (2)$$

Properties of the closure operator

Definition (Reflecting Zero)

A morphism $X \xrightarrow{f} Y$ is said to *reflect zero* if $f^{-1}\sigma_Y = \sigma_X$.

Properties of the closure operator

The following three statements are equivalent for any morphism $X \xrightarrow{f} Y$:

(a) f reflects zero

(b) For each $x \in \mathtt{Sub}_{\mathsf{M}}(X)$, $\exists_f x = \sigma_Y \Rightarrow x = \sigma_X$

(c) For each $x \in \mathtt{Sub}_{\mathsf{M}}(X)$ and $y \in \mathtt{Sub}_{\mathsf{M}}(Y)$, $y \wedge \exists_f x = \sigma_Y \Rightarrow x \wedge f^{-1}y = \sigma_X$

see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Theorem 9.2

Properties of the closure operator

In any context if every morphism reflects zero then the initial object $\emptyset$ is strict. Conversely, if the initial object $\emptyset$ is strict and the unique morphism $\emptyset \to 1$ is an admissible monomorphism then every morphism reflects zero.

see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Theorem 9.2

Properties of the closure operator

A category is said to be *quasi-pointed* (see Bourn, "3×3 lemma and protomodularity", §1, and see Goswami and Janelidze, "On the structure of zero morphisms in a quasi-pointed category") if the unique morphism $\emptyset \to 1$ is a monomorphism.

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A context shall be called *admissibly quasi-pointed* if the unique morphism $\emptyset \to 1$ is an admissible monomorphism.

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Several contexts are admissibly quasi-pointed — e.g., sets and functions, topological spaces and continuous maps, locales and localic maps, where this unique morphism is a regular monomorphism and hence admissible; however the context of rings and their homomorphisms is **not** quasi-pointed even.

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(b) For each $x \in \mathtt{Sub}_{\mathsf{M}}(X)$, $\exists_f x = \sigma_Y \Rightarrow x = \sigma_X$

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Conversely, if the initial object $\emptyset$ is strict and the unique morphism $\emptyset \to 1$ is an admissible monomorphism then every morphism reflects zero.

see ibid., Theorem 9.2

Thus: in admissibly quasi-pointed contexts, the initial object is strict if and only if every morphism reflects zero.

Properties of the closure operator

Theorem (Properties of Closure, see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Theorem 3.1)

- *If every filter of* $\mathtt{Sub}_{\mathsf{M}}(X)$ *is contained in a prime filter then* cl_μ *is additive, i.e., for each* $x, y \in \mathtt{Sub}_{\mathsf{M}}(X)$*:*

$$\mathrm{cl}_\mu(x \vee y) = \mathrm{cl}_\mu x \vee \mathrm{cl}_\mu y. \tag{2}$$

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Even in the presence of **Axiom of Choice** the presence of a prime filter in a complete non-distributive lattice is not guaranteed.

Absence of distributivity in the lattice does not even ensure a maximal filter to be prime.

See Erné, "Prime and maximal ideals of partially ordered sets" for details.

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- *Given any internal preneighbourhood space* (X, μ)*, the function* $\mathtt{Sub}_{\mathsf{M}}(X) \xrightarrow{\mathrm{cl}_\mu} \mathtt{Sub}_{\mathsf{M}}(X)$ *defines a closure operation on* $\mathtt{Sub}_{\mathsf{M}}(X)$ *such that* $\mathrm{cl}_\mu \sigma_X = \sigma_Y$*.*
- *If* $(X, \mu) \xrightarrow{f} (Y, \phi)$ *is a preneighbourhood morphism reflecting zero then it is* μ-ϕ continuous*, i.e., for any* $p \in \mathtt{Sub}_{\mathsf{M}}(X)$*:*

$$\exists_f \mathrm{cl}_\mu p \leq \mathrm{cl}_\phi \exists_f p \tag{2}$$

- *If every filter of* $\mathtt{Sub}_{\mathsf{M}}(X)$ *is contained in a prime filter then* cl_μ *is* additive*, i.e., for each* $x, y \in \mathtt{Sub}_{\mathsf{M}}(X)$*:*

$$\mathrm{cl}_\mu(x \vee y) = \mathrm{cl}_\mu x \vee \mathrm{cl}_\mu y. \tag{3}$$

- *The closure operation is* hereditary*, i.e., given* $A \xrightarrow{a} M \xrightarrow{m} X$ *,*

$$\mathrm{cl}_{(\mu|_m)} a = m^{-1}(\mathrm{cl}_\mu(m \circ a));$$

hence a *and* m *closed imply* $m \circ a$ *closed.*

Proper Morphisms

Definition (see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Definition 6.1)

A preneighbourhood morphism $(X,\mu) \xrightarrow{f} (Y,\phi)$ is *proper* if for every preneighbourhood morphism $(Z,\psi) \xrightarrow{g} (Y,\phi)$ and pullback

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{f_g} & Z \\ {\scriptstyle g_f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

, the morphism $(X \times_Y Z, \mu \times_\phi \psi) \xrightarrow{f_g} (Z,\psi)$ is a closed morphism.

The set $\mathbb{A}_{\text{pr}}$ is the (possibly large) set of all proper morphisms.

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$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{f_g} & Z \\ {\scriptstyle g_f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

, the morphism $(X \times_Y Z, \mu \times_\phi \psi) \xrightarrow{f_g} (Z, \psi)$ is a closed morphism.

The set $\mathbb{A}_{\mathsf{pr}}$ is the (possibly large) set of all proper morphisms.

Examples
$(\mathtt{Set}, \mathtt{Sur}, \mathtt{Inj})$
$(\mathtt{Top}, \mathtt{Epi}, \mathtt{ExtMon})$
$(\mathtt{Loc}, \mathtt{Epi}, \mathtt{RegMon})$

Proper Morphisms

Definition (see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Definition 6.1)

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$(\mathtt{Loc}, \mathtt{Epi}, \mathtt{RegMon})$	for locales with T-neighbourhood systems, usual proper maps of locales

Proper Morphisms

The *T-neighbourhood system* was investigated in the papers Dube and Ighedo, "More on locales in which every open sublocale is z-embedded"; Dube and Ighedo, "Characterising points which make P-frames", christened in Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms" and for any locale X, it is the order preserving map $\mathtt{Sub}_{\mathtt{RegMon}}(X)^{\mathrm{op}} \xrightarrow{\mathfrak{o}_X} \mathtt{Fil}X$ defined by:

$$\mathfrak{o}_X(S) = \{T \in \mathtt{Sub}_{\mathtt{RegMon}}(X) : (\exists a \in X)(S \subseteq \mathfrak{O}[a] \subseteq T)\}.$$

It is a neighbourhood system on X, and the functor with object function $X \mapsto (X, \mathfrak{o}_X)$ is right inverse to the forgetful functor $\mathtt{pNbd}[\mathtt{Loc}] \xrightarrow{U} \mathtt{Loc}$ (see Ghosh, "Internal neighbourhood structures", Theorem 3.38)

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Proper Morphisms

Theorem (Alternative characaterisation of proper morphisms, see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Theorem 6.1(a))

A preneighbourhood morphism $(X,\mu) \xrightarrow{f} (Y,\phi)$ *is proper if and only if for every preneighbourhood space* (Z,ψ)*, every corestriction of* $(X \times Z, \mu \times \psi) \xrightarrow{f \times 1_Z} (Y \times Z, \phi \times \psi)$ *is a closed morphism.*

Separated Morphisms

Lemma (see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Lemma 7.1)

If $(X, \mu) \xrightarrow{f} (Y, \phi)$ *is a preneighbourhood morphism,*

$$\begin{array}{ccc} \mathtt{kerp}\, f & \xrightarrow{p_2} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

be its kernel pair, $X \xrightarrow{d_f = (1_X, 1_X)} \mathtt{kerp}\, f$ *be the diagonal morphism, then* $\mu = ((\mu \times_\phi \mu)\big|_X)$ *and* d_f *is an embedding.*

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Lemma (see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Lemma 7.1)

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Definition (see ibid., Definition 7.1)

A preneighbourhood morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ is *separated* if d_f is a proper morphism.

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A preneighbourhood morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ is *separated* if d_f is a proper morphism.

If every preneighbourhood morphism is continuous then f is separated if and only if d_f is a closed embedding — compare with the notion of *separated morphisms* in Clementino, Giuli, and Tholen, "A functional approach to general topology".

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Examples	
$(\mathtt{Set}, \mathtt{Sur}, \mathtt{Inj})$	for internal neighbourhood spaces, continuous maps in whose fibres distinct points have disjoint neighbourhoods
$(\mathtt{Top}, \mathtt{Epi}, \mathtt{ExtMon})$	for internal neighbourhood spaces, separated maps between the second topology

Properties of closed, proper and separated morphisms

Summary of properties of closed/proper/separated morphisms, given the preneighbourhood morphisms $(X, \mu) \xrightarrow{f} (Y, \phi) \xrightarrow{g} (Z, \psi)$

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$\mathbb{A}_{\mathrm{cl}}$		
contain isomorphisms		
closed under compositions		
$g \circ f$ is closed, f is a continuous formal surjection imply g is closed		
if $m \in \mathfrak{C}_\phi$ then f is continuous implies $f^{-1}m \in \mathfrak{C}_\mu$, f is closed and continuous imply f_m is closed		

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$\mathbb{A}_{\mathsf{cl}}$	$\mathbb{A}_{\mathsf{pr}}$	
contain isomorphisms	contain closed embeddings in a reflecting zero context	
closed under compositions	closed under compositions	
$g \circ f$ is closed, f is a continuous formal surjection imply g is closed	$g \circ f$ is proper, f is continuously stably in E imply g is proper	
	$g \circ f$ is proper, g is a monomorphism imply f is proper	
if $m \in \mathfrak{C}_\phi$ then f is continuous implies $f^{-1}m \in \mathfrak{C}_\mu$, f is closed and continuous imply f_m is closed	pullback stable	

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A morphism $X \xrightarrow{f} Y$ is a *formal surjection* if $y \in \mathtt{Sub}_{\mathsf{M}}(Y) \Rightarrow (\exists x \in \mathtt{Sub}_{\mathsf{M}}(X))(y = \exists_f x)$, or equivalently, for each $y \in \mathtt{Sub}_{\mathsf{M}}(Y)$, $f_y \in \mathsf{E}$.

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Summary of properties of closed/proper/separated morphisms, given the preneighbourhood morphisms $(X,\mu) \xrightarrow{f} (Y,\phi) \xrightarrow{g} (Z,\psi)$

	$g \circ f$ is proper, f is continuously stably in E imply g is proper	$g \circ f$ is separated, f is proper and continuously stably in E imply g is separated

A preneighbourhood morphism $(X,\mu) \xrightarrow{f} (Y,\phi)$ is *continuously stably in E* if for every preneighbourhood morphism $(Z,\psi) \xrightarrow{g} (Y,\phi)$, the pullback f_g of f along g is $((\mu \times_\phi \psi), \psi)$-continuous and is in E.

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Summary of properties of closed/proper/separated morphisms, given the preneighbourhood morphisms $(X,\mu) \xrightarrow{f} (Y,\phi) \xrightarrow{g} (Z,\psi)$

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if $m \in \mathfrak{C}_\phi$ then f is continuous implies $f^{-1}m \in \mathfrak{C}_\mu$, f is closed and continuous imply f_m is closed	pullback stable	pullback stable

Compare the properties for similar morphisms in Clementino, Giuli, and Tholen, "A functional approach to general topology", where *continuous* condition is automatic.

Hausdorff preneighbourhood spaces

The smallest preneighbourhood system on an object X is $\mathtt{Sub}_{\mathsf{M}}(X)^{\mathrm{op}} \xrightarrow{\nabla_X} \mathtt{Fil}X$, where:

$$\nabla_X(p) = \begin{cases} \mathtt{Sub}_{\mathsf{M}}(X), & \text{if } p = \sigma_X \\ \{1_X\}, & \text{if } p \neq \sigma_X \end{cases}.$$

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The terminal object 1 being the empty product is considered as an internal preneighbourhood space with its smallest preneighbourhood system ∇_1.

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Definition (see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", §7.2)

An internal preneighbourhood space (X, μ) is said to be *Hausdorff* if the unique morphism $(X, \mu) \xrightarrow{t_X} (1, \nabla_1)$ is separated.

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In $(\mathtt{Set}, \mathtt{Surjection}, \mathtt{Injection})$ the terminal object is singleton, $\nabla_1 = \uparrow_1$.

However, in $(\mathtt{CRing}^{\mathrm{op}}, \mathtt{Epi}, \mathtt{RegMon})$ the terminal object is $\mathbb{Z}$ and $\nabla_1 < \uparrow_1$.

Examples of Hausdorff preneighbourhood spaces

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- $(\mathtt{Grp}, \mathtt{RegEpi}, \mathtt{Mon})$ every Hausdorff topological group is a Hausdorff neighbourhood space
- $(\mathtt{Loc}, \mathtt{Epi}, \mathtt{RegMon})$ every Hausdorff locale is a Hausdorff preneighbourhood space with its functorial T-neighbourhood system

Alternative characterisation of Hausdorff preneighbourhood spaces

Theorem (see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Theorem 7.3)

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- *If* $(X \times Y, \mu \times \phi) \xrightarrow{p_2} (Y, \phi)$ *is the product projection then* p_2 *is separated.*
- *The product internal preneighbourhood space* $(X \times Y, \mu \times \phi)$ *is Hausdorff whenever* (Y, ϕ) *is Hausdorff.*

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- *If* $\left(E,(\psi|_E)\right)\xrightarrow{e}(Z,\psi)\overset{f}{\underset{g}{\rightrightarrows}}(X,\mu)$ *is an equaliser diagram then* e *is a proper morphism.*

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- *Every preneighbourhood morphism with (X, μ) as domain is separated.*
- *There exists a separated preneighbourhood morphism with domain (X, μ) and codomain a Hausdorff preneighbourhood space (Y, ϕ).*
- *If $(X, \mu) \xrightarrow{f} (Y, \phi)$ is a proper morphism stably continuously in E then (Y, ϕ) is Hausdorff.*
- *If $(X \times Y, \mu \times \phi) \xrightarrow{p_2} (Y, \phi)$ is the product projection then p_2 is separated.*
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- *If $\left(E, (\psi|_E)\right) \xrightarrow{e} (Z, \psi) \overset{f}{\underset{g}{\rightrightarrows}} (X, \mu)$ is an equaliser diagram then e is a proper morphism.*

Compare similar characterisations of Hausdorffness Clementino, Giuli, and Tholen, "A functional approach to general topology", especially where *continuous* condition is automatic.

The category of Hausdorff preneighbourhood spaces

$\mathtt{Haus}[\mathbb{A}]$ is the full subcategory of Hausdorff preneighbourhood spaces.

The category of Hausdorff preneighbourhood spaces

- $\mathtt{Haus}[\mathbb{A}]$ is finitely complete, closed under subobjects and images of morphisms stably continuously in E.

see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Corollary 7.4

The category of Hausdorff preneighbourhood spaces

- $\mathtt{Haus}[\mathbb{A}]$ is finitely complete, closed under subobjects and images of morphisms stably continuously in E.
see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Corollary 7.4
- In an extensive context with finite sum of closed morphisms closed, $\mathtt{Haus}[\mathbb{A}]$ is closed under finite sums if and only if $(1+1, \nabla_1 + \nabla_1)$ is Hausdorff.
see Ghosh, "Internal neighbourhood structures III: Finite sum of subobjects", Theorem 10.1

The category of Hausdorff preneighbourhood spaces

- A category with finite sums is *extensive* if the sum functor $(\mathbb{A}\downarrow A)\times(\mathbb{A}\downarrow B)\xrightarrow{+}(\mathbb{A}\downarrow A+B)$ is an equivalence of categories (see Carboni, Lack, and Walters, "Introduction to extensive and distributive categories", for details...).
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In short, these are precisely categories where sums behave well with pullbacks.
- In an extensive context, a finite sum of admissible subobjects is an admissible subobject if and only if the monomorphisms in E between finite sums are stable under pullbacks along coproduct injections.
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- In an extensive context, a finite sum of admissible subobjects is an admissible subobject if and only if the monomorphisms in E between finite sums are stable under pullbacks along coproduct injections.
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- In an extensive context the following statements are equivalent:

(a) Every finite sum of closed embeddings is a closed embedding.

(b) Every finite sum of admissible subobjects is an admissible subobject and each coproduct injection is a closed embedding.

(c) Each *dense* morphism between finite sums is stable under pullbacks along coproduct injections.

In particular, in an extensive context in which finite sum of admissible subobjects is admissible, a finite sum of closed embeddings is a closed embedding if and only if the coproduct injections are closed.
see ibid., Theorem 5.1, Corollary 5.2

The category of Hausdorff preneighbourhood spaces

$\mathtt{Haus}[\mathbb{A}]$ is the full subcategory of Hausdorff preneighbourhood spaces.

- $\mathtt{Haus}[\mathbb{A}]$ is finitely complete, closed under subobjects and images of morphisms stably continuously in E.

see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Corollary 7.4

- In an extensive context with finite sum of closed morphisms closed, $\mathtt{Haus}[\mathbb{A}]$ is closed under finite sums if and only if $(1+1, \nabla_1 + \nabla_1)$ is Hausdorff.

see Ghosh, "Internal neighbourhood structures III: Finite sum of subobjects", Theorem 10.1

Hausdorff reflection

Theorem (see Ghosh, "Internal neighbourhood structures IV: Internal Hausdorff Spaces", Theorem 3.2)

$\mathtt{Haus}[\mathbb{A}]$ *is (regular epi)-reflective subcategory of* $\mathtt{pNbd}[\mathbb{A}]$*, provided the product projections are in* $\mathtt{E}$ *and every morphism reflects zero.*

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Let $X \xrightarrow{\mathfrak{h}} \left[\frac{X}{R}\right]$ be the coequaliser of the pair $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$.

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Then: $\left(\left[\frac{X}{R}\right], \mu_{\mathfrak{h}}\right)$ is a Hausdorff preneighbourhood space, $(X, \mu) \xrightarrow{\mathfrak{h}} \left(\left[\frac{X}{R}\right], \mu_{\mathfrak{h}}\right)$ the required reflection.

Hausdorff reflection

Theorem (see Ghosh, "Internal neighbourhood structures IV: Internal Hausdorff Spaces", Theorem 3.2)

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Remarks on the proof:

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Remarks on the proof:

- Assumption of every morphism reflecting zero simplifies the description of Hausdorffness — it is enough to check d_X is closed.

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- The subobject lattice $\mathtt{Sub}_{\mathsf{M}}(X)$ is *large*, and hence has *large* meets/joins. Extending the set theoretic universe accommodating *conglomerates* as in Adámek, Herrlich, and Strecker, *Abstract and concrete categories* explains the (possibly large) join for the subobject (r_1, r_2).

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- Since $\mathtt{pNbd}[\mathbb{A}] \underset{\text{incl}}{\overset{\mathfrak{h}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathtt{Haus}[\mathbb{A}]$ and $\mathtt{pNbd}[\mathbb{A}]$ is topological over $\mathbb{A}$, $\mathtt{Haus}[\mathbb{A}]$ is as (co)complete as the category $\mathbb{A}$.

Hausdorff reflection

In what follows the Hausdorff reflection functor is:

$$\mathfrak{h} : \left. \begin{array}{ccc} \texttt{pNbd}[\mathbb{A}] & \longrightarrow & \texttt{Haus}[\mathbb{A}] \\ & & \\ (X,\mu) & \longmapsto & \left(\mathfrak{h}X, \mu_{\mathfrak{h}}\right) \\ f \downarrow & \longmapsto & \downarrow \mathfrak{h}f \\ (Y,\phi) & \longmapsto & \left(\mathfrak{h}Y, \phi_{\mathfrak{h}}\right) \end{array} \right\}$$

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where $\left(\mathfrak{h}X, \mu_{\mathfrak{h}}\right) \xrightarrow{\mathfrak{h}f} \left(\mathfrak{h}Y, \phi_{\mathfrak{h}}\right)$ is the unique preneighbourhood morphism such that the diagram:

$$\begin{array}{ccc} X & \xrightarrow{h_X} & \mathfrak{h}X \\ f \Big\downarrow & & \Big\downarrow \mathfrak{h}f \\ Y & \xrightarrow[h_Y]{} & \mathfrak{h}Y \end{array}$$

commutes.

Hausdorff reflection

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$$\begin{array}{ccccc}
\texttt{kerp}\, h_X & \overset{h_{X,1}}{\underset{h_{X,2}}{\rightrightarrows}} & X & \overset{h_X}{\twoheadrightarrow} & \mathfrak{h}X \\
 & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathfrak{h}f} \\
\texttt{kerp}\, h_Y & \overset{h_{Y,1}}{\underset{h_{Y,2}}{\rightrightarrows}} & Y & \underset{h_Y}{\twoheadrightarrow} & \mathfrak{h}Y
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commutes.

Since h_X is a regular epimorphism it is the coequaliser of its kernel pair as shown on the left.

Hausdorff reflection

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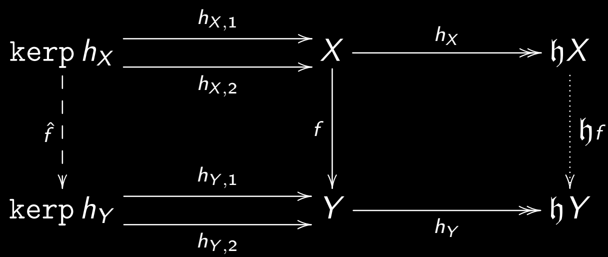


commutes.

Since h_X is a regular epimorphism it is the coequaliser of its kernel pair as shown on the left.

Since $h_Y \circ f \circ h_{X,1} = h_Y \circ f \circ h_{X,2}$, there exists the unique morphism $\hat{f}$ such that the squares on the left reasonably commutes, i.e., $f \circ h_{X,i} = h_{Y,i} \circ \hat{f}$, $i = 1, 2$.

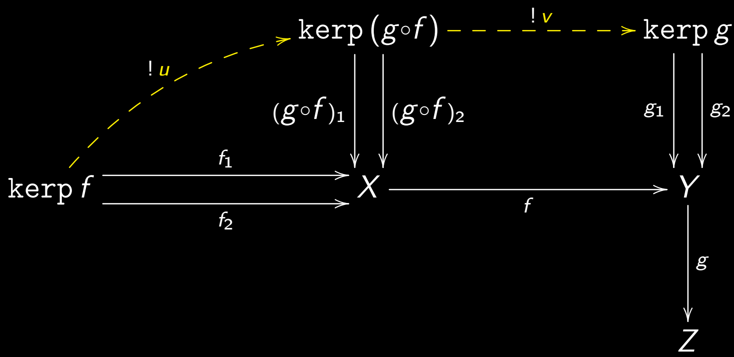
Transfinite construction of Hausdorff reflection

Theorem (see Ghosh, "Internal neighbourhood structures IV: Internal Hausdorff Spaces", Theorem 4.1)

In a reflecting zero context with each finite product projection in E, if $(X,\mu) \xrightarrow{f} (Y,\phi)$ *is a preneighbourhood morphism with codomain a Hausdorff preneighbourhood space then* $\mathtt{kerp}\, f$ *is closed in* $(X \times X, \mu \times \mu)$*.*

In particular, for the Hausdorff reflection $(X,\mu) \xrightarrow{h_X} \left(\mathfrak{h}X, \mu_{\mathfrak{h}}\right)$, $\mathtt{kerp}\, h_X$ *is the smallest internal equivalence relation on* X *such that its quotient preneighbourhood space is Hausdorff.*

Transfinite construction of Hausdorff reflection



Evidently, $\mathtt{kerp}\, g \leq d_Y \Leftrightarrow g_1 = g_2 \Rightarrow u \in \mathtt{Iso}(\mathbb{A})$.
If v is an epimorphism then $u \in \mathtt{Iso}(\mathbb{A}) \Rightarrow \mathtt{kerp}\, g \leq d_Y$.
Proof:
$g_1 \circ v = f \circ f_1 \circ u^{-1} = f \circ f_2 \circ u^{-1} = g_2 \circ v$ completes the proof.

Transfinite construction of Hausdorff reflection

Transfinite Construction:

Let (X, μ) be an internal preneighbourhood space.

Transfinite construction of Hausdorff reflection

- Step 1:

Take $q_0 = 1_X$.
Then: $\mathtt{kerp}\, q_0 = d_X \leq \mathtt{kerp}\, h_X$.

Transfinite construction of Hausdorff reflection

- Step 2:

Assume α is a non-limit ordinal, $\alpha = \beta + 1$ and for each $\gamma \leq \beta$, q_γ is defined, $\mathtt{kerp}\, q_\gamma \leq \mathtt{kerp}\, h_X$ and $\gamma \leq \gamma' \leq \beta \Rightarrow \mathtt{kerp}\, q_\gamma \leq \mathtt{kerp}\, q_{\gamma'}$. Consider the diagram:

$$\mathtt{kerp}\, q_\beta \underset{q_{\beta,2}}{\overset{q_{\beta,1}}{\rightrightarrows}} X \xrightarrow{\;q_\beta\;} Y_\beta$$

Transfinite construction of Hausdorff reflection

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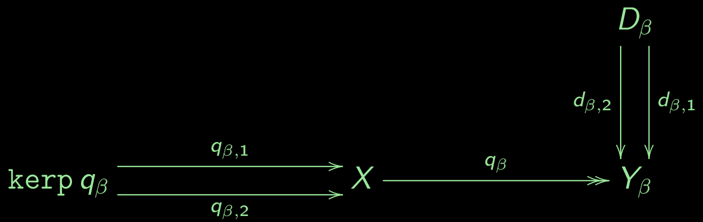

Take: μ_β quotient preneighbourhood system on Y_β, $D_\beta \overset{d_{\beta,1}}{\underset{d_{\beta,2}}{\rightrightarrows}} Y_\beta = \mathrm{cl}_{\mu_\beta} d_{Y_\beta}$.

Transfinite construction of Hausdorff reflection

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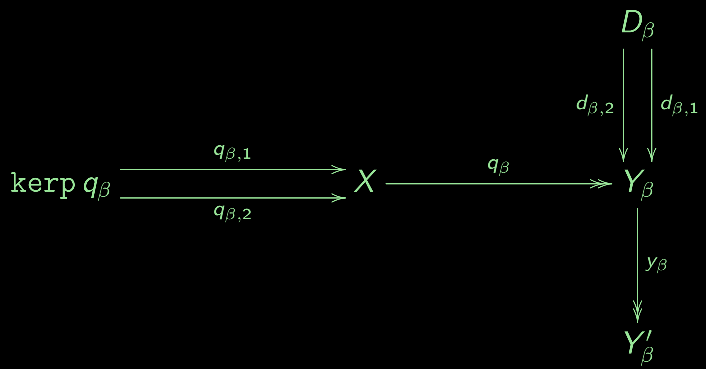

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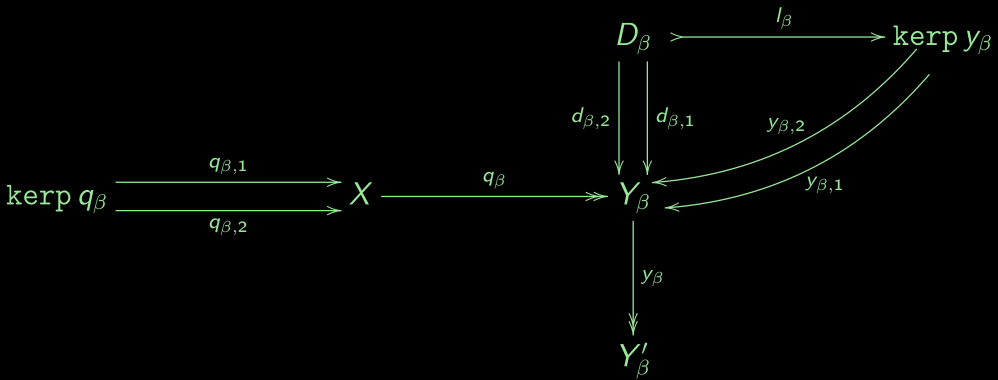

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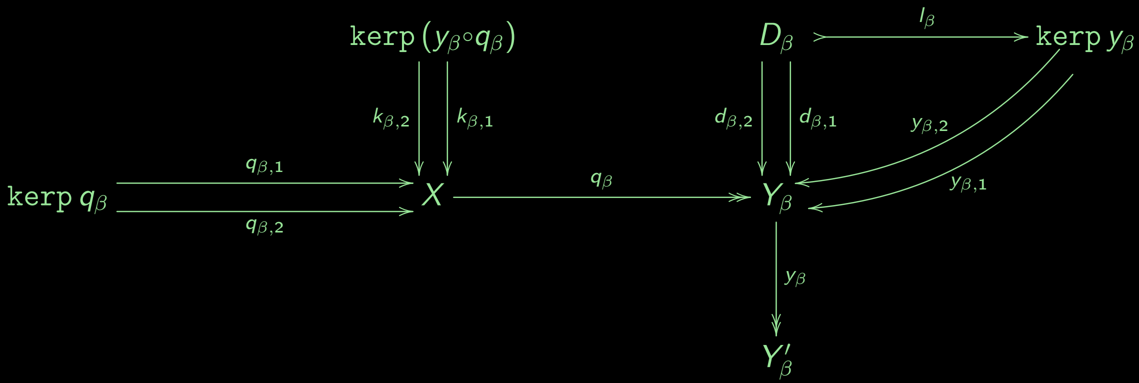

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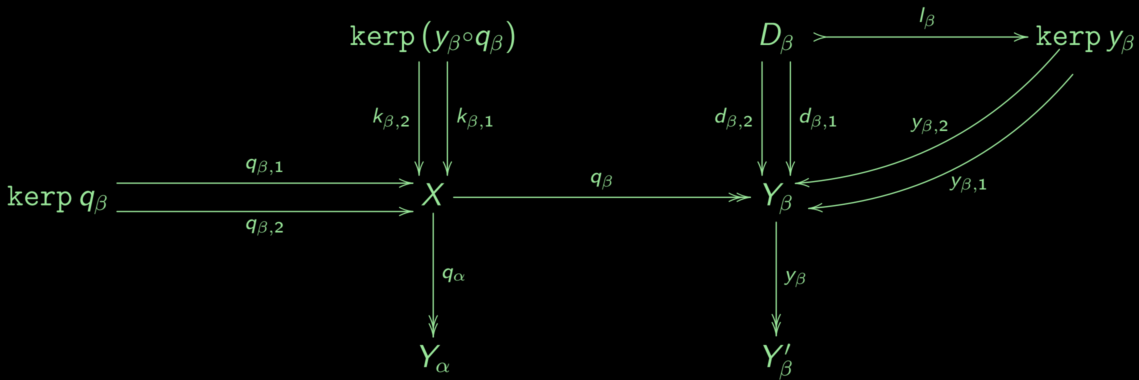

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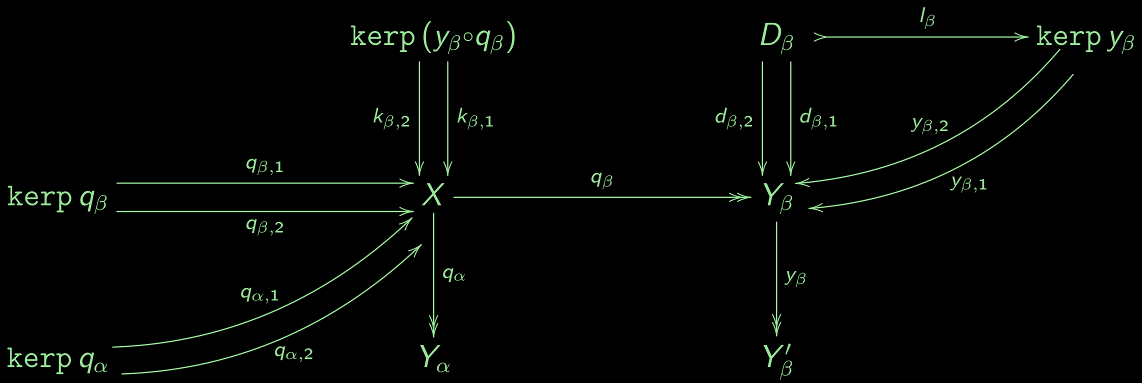

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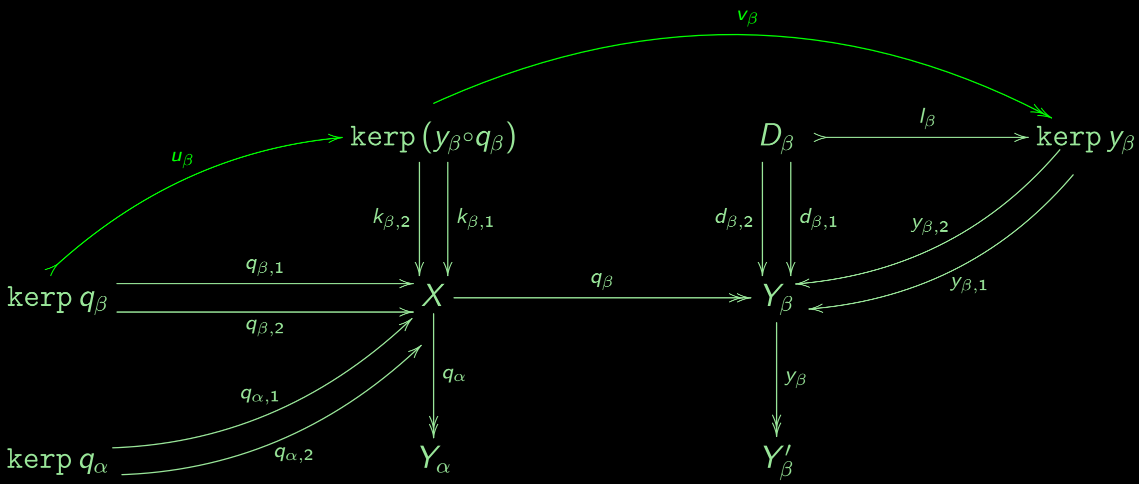

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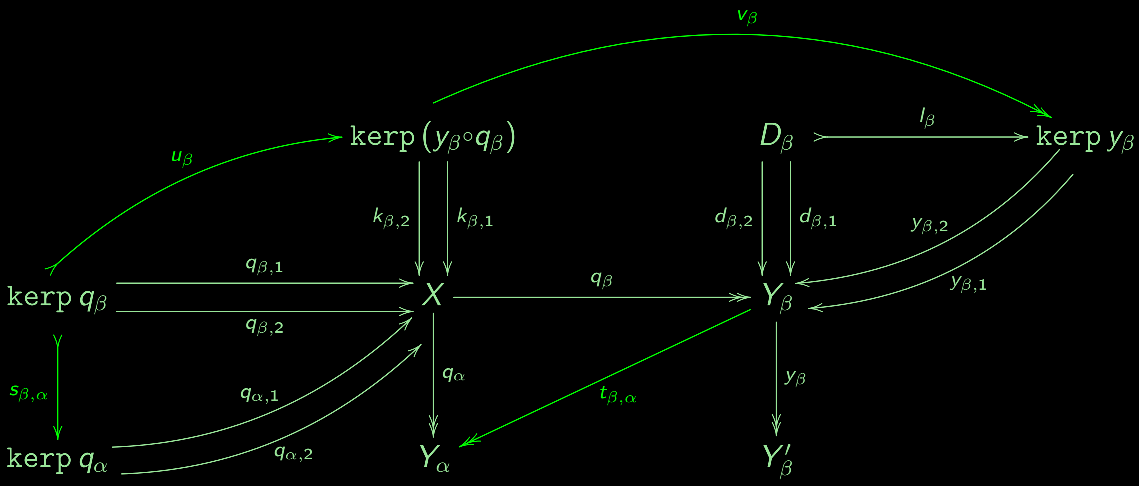

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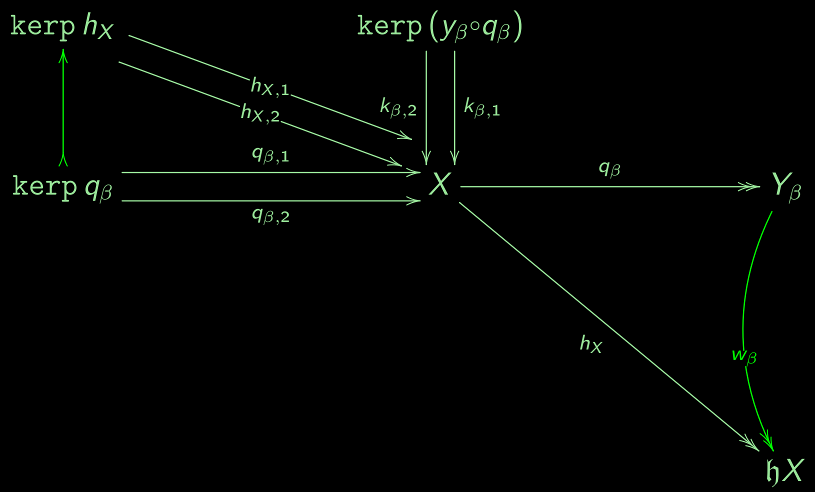

Since $\mathtt{kerp}\, q_\beta \leq \mathtt{kerp}\, h_X$ there exists the morphism w_β such that $h_X = w_\beta \circ q_\beta$.

Transfinite construction of Hausdorff reflection

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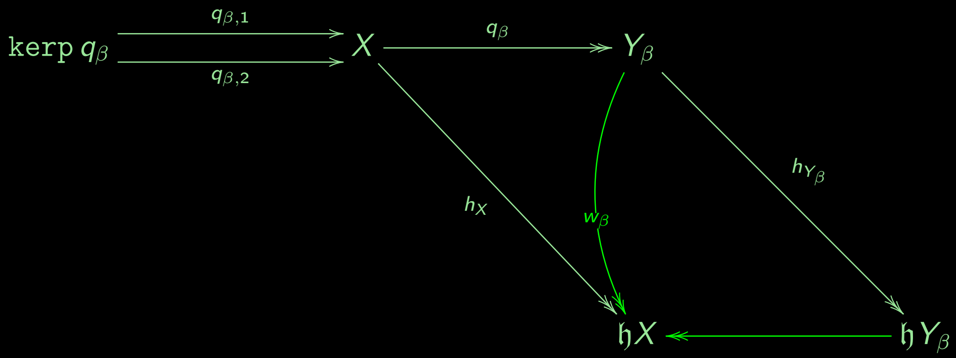

Since $\mathfrak{h}X$ is Hausdorff, w_β factors through the Hausdorff reflection h_{Y_β}.

Transfinite construction of Hausdorff reflection

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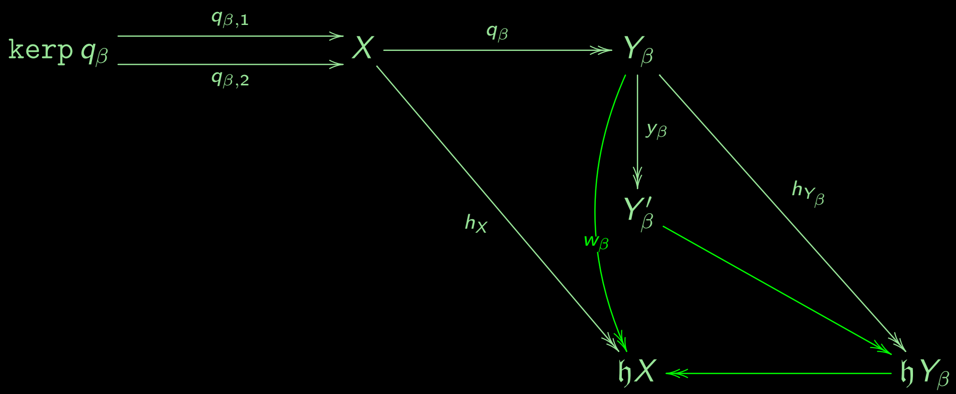

Since $(d_{\beta,1}, d_{\beta,2}) \leq \mathtt{kerp}\, h_{Y_\beta}$, h_{Y_β} factors through the coequaliser y_β.

Transfinite construction of Hausdorff reflection

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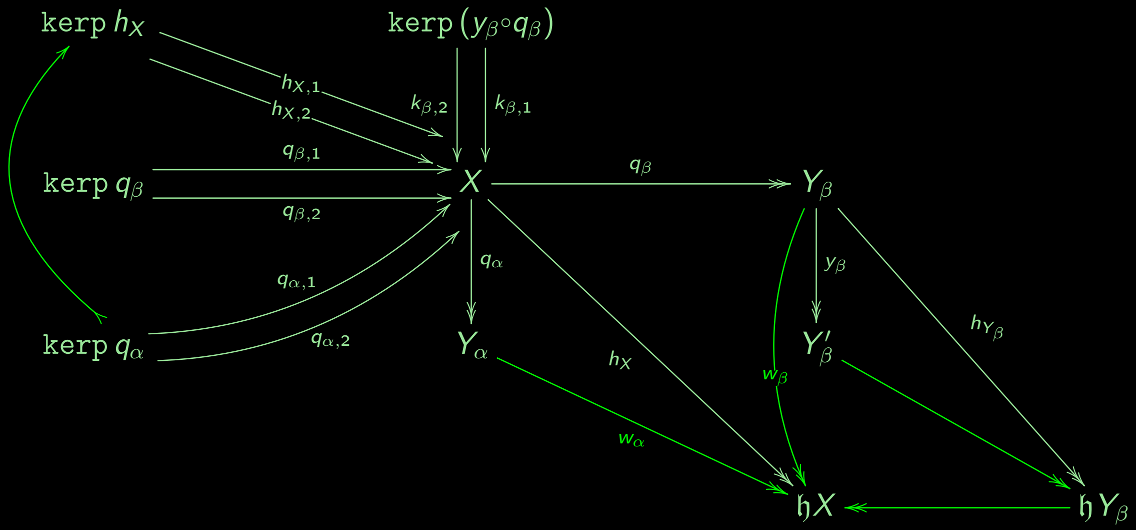
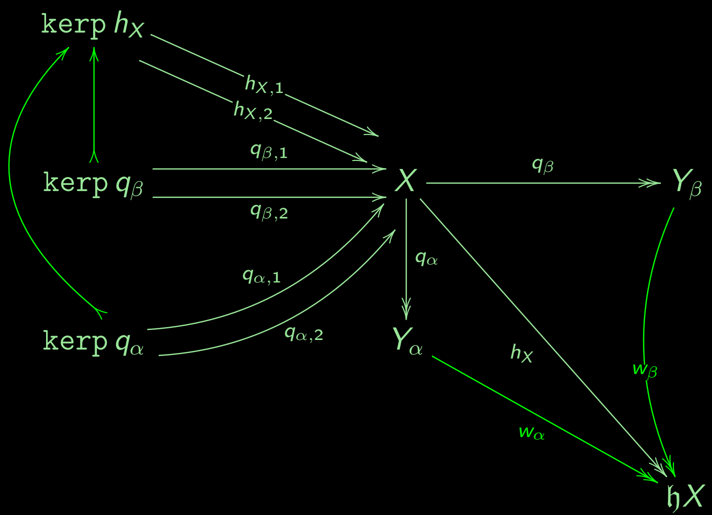


Since w_β factors through y_β, $h_X \circ k_{\beta,1} = w_\beta \circ q_\beta \circ k_{\beta,1} = w_\beta \circ q_\beta \circ k_{\beta,2} = h_X \circ k_{\beta_2}$.

Transfinite construction of Hausdorff reflection

- Step 2:

Assume α is a non-limit ordinal, $\alpha = \beta + 1$ and for each $\gamma \leq \beta$, q_γ is defined, $\operatorname{kerp} q_\gamma \leq \operatorname{kerp} h_X$ and $\gamma \leq \gamma' \leq \beta \Rightarrow \operatorname{kerp} q_\gamma \leq \operatorname{kerp} q_{\gamma'}$. Consider the diagram:



Hence $\operatorname{kerp} q_\alpha \leq \operatorname{kerp} h_X$.

Transfinite construction of Hausdorff reflection

- Step 3:

Assume α is a limit ordinal and for each $\beta < \alpha$, q_β is defined, $\mathtt{kerp}\, q_\beta \leq \mathtt{kerp}\, h_X$ and $\gamma < \gamma' < \alpha \Rightarrow \mathtt{kerp}\, q_\gamma \leq \mathtt{kerp}\, q_{\gamma'}$. Consider the diagram:

$$\mathtt{kerp}\, q_\beta \underset{q_{\beta,2}}{\overset{q_{\beta,1}}{\rightrightarrows}} X \overset{q_\beta}{\twoheadrightarrow} Y_\beta$$

Transfinite construction of Hausdorff reflection

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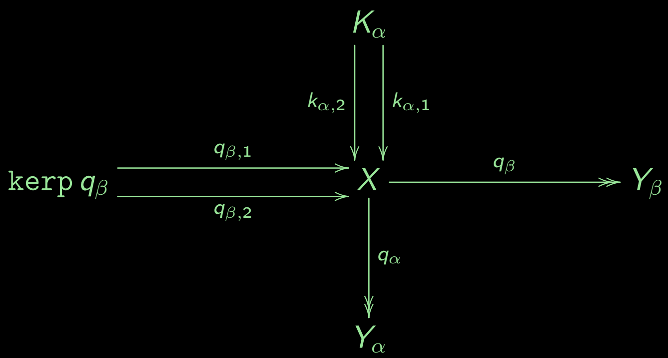

Take $K_\alpha \overset{k_{\alpha,1}}{\underset{k_{\alpha,2}}{\rightrightarrows}} X = \bigvee_{\beta<\alpha} \mathtt{kerp}\, q_\beta$, q_α is the coequaliser of the pair $(k_{\alpha,1}, k_{\alpha,2})$, μ_α is the quotient preneighbourhood system on Y_α.

Transfinite construction of Hausdorff reflection

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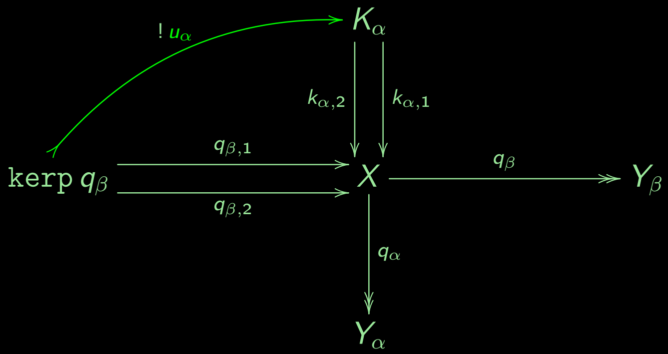
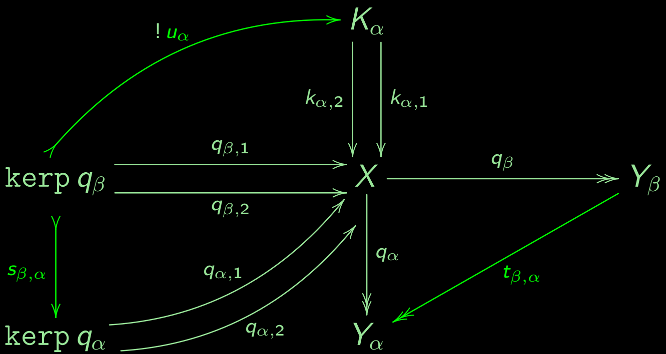


Since $\mathtt{kerp}\, q_\beta \leq K_\alpha$, there exists the morphism u_α such that $q_{\beta,i} = k_{\alpha,i} \circ u_\alpha$ for each $\beta < \alpha$.

Transfinite construction of Hausdorff reflection

- Step 3:

Assume α is a limit ordinal and for each $\beta < \alpha$, q_β is defined, $\mathtt{kerp}\, q_\beta \leq \mathtt{kerp}\, h_X$ and $\gamma < \gamma' < \alpha \Rightarrow \mathtt{kerp}\, q_\gamma \leq \mathtt{kerp}\, q_{\gamma'}$. Consider the diagram:

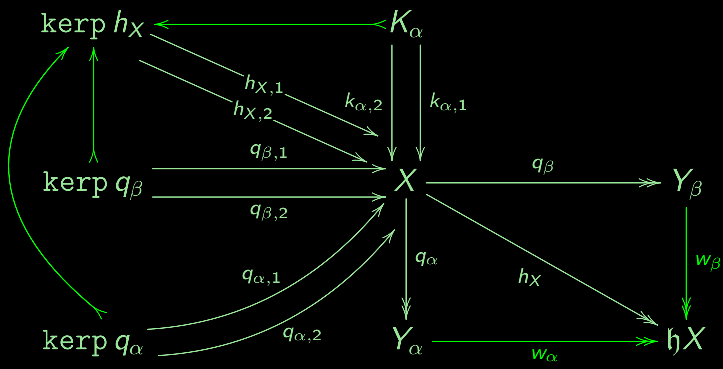


Hence there exists the unique morphism $s_{\beta,\alpha}$ and $t_{\beta,\alpha}$ such that $q_{\beta,i} = q_{\alpha,i} \circ s_{\beta,\alpha}$ and $q_\alpha = t_{\beta,\alpha} \circ q_\beta$ for each $\beta < \alpha$ and $i = 1, 2$.

Transfinite construction of Hausdorff reflection

- Step 3:

Assume α is a limit ordinal and for each $\beta < \alpha$, q_β is defined, $\mathtt{kerp}\, q_\beta \leq \mathtt{kerp}\, h_X$ and $\gamma < \gamma' < \alpha \Rightarrow \mathtt{kerp}\, q_\gamma \leq \mathtt{kerp}\, q_{\gamma'}$. Consider the diagram:

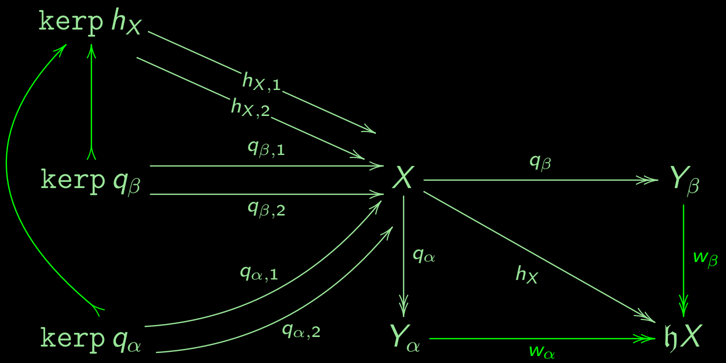


Since $\mathtt{kerp}\, q_\beta \leq \mathtt{kerp}\, h_X$ there exists the morphism w_β such that $h_X = w_\beta \circ q_\beta$. Hence $(k_{\alpha,1}, k_{\alpha,2}) = \bigvee_{\beta<\alpha} \mathtt{kerp}\, q_\beta \leq \mathtt{kerp}\, h_X$.

Transfinite construction of Hausdorff reflection

- Step 3:

Assume α is a limit ordinal and for each $\beta < \alpha$, q_β is defined, $\texttt{kerp}\, q_\beta \leq \texttt{kerp}\, h_X$ and $\gamma < \gamma' < \alpha \Rightarrow \texttt{kerp}\, q_\gamma \leq \texttt{kerp}\, q_{\gamma'}$. Consider the diagram:



Hence $\texttt{kerp}\, q_\alpha \leq \texttt{kerp}\, h_X$.

Transfinite construction of Hausdorff reflection

Transfinite Construction:

Let (X, μ) be an internal preneighbourhood space.

There exists a transfinite sequence $\langle\, (X,\mu) \xrightarrow{q_\alpha} (Y_\alpha, \mu_\alpha) \;:\alpha$ is an ordinal$\rangle$ such that:

(a) μ_α is the largest preneighbourhood system on Y_α such that q_α is a preneighbourhood morphism, and

(b) $0 \le \beta \le \alpha \Rightarrow \mathtt{kerp}\, q_\beta \le \mathtt{kerp}\, q_\alpha \le \mathtt{kerp}\, h_X$.

Transfinite construction of Hausdorff reflection

Theorem (see Ghosh, "Internal neighbourhood structures IV: Internal Hausdorff Spaces", Theorem 4.2)

In every reflecting zero context with finite product projections in E and stable regular epimorphisms, there exists a transfinite construction of the Hausdorff reflection of an internal preneighbourhood space (X, μ) *for which* $\mathtt{Sub}_{\mathsf{M}}(X \times X)$ *is a small set.*

Transfinite construction of Hausdorff reflection

Sketch of proof:

Take the transfinite sequence $\langle\, (X, \mu) \xrightarrow{q_\alpha} (Y_\alpha, \mu_\alpha)\ : \alpha$ is an ordinal$\rangle$.

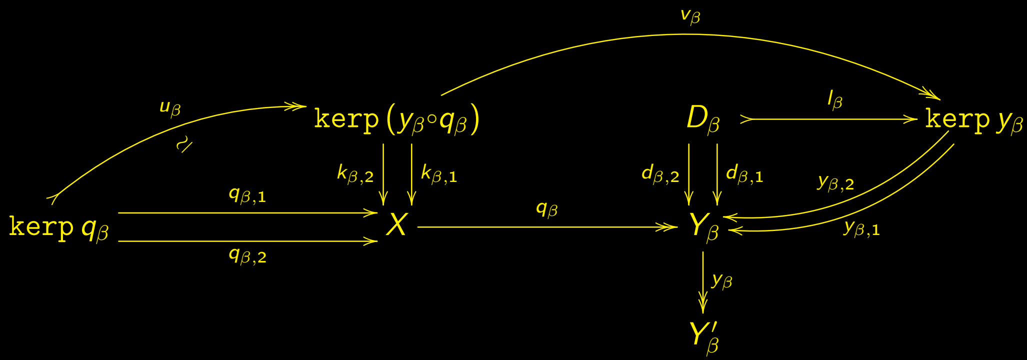
Since $\mathtt{Sub}_{\mathsf{M}}(X \times X)$ is a small set, there exists an ordinal β such that $\mathtt{kerp}\, q_\beta = \mathtt{kerp}\, q_{\beta+1}$.

Hence u_β is an isomorphism.

Transfinite construction of Hausdorff reflection

Sketch of proof:

Take the transfinite sequence $\langle\ (X,\mu) \xrightarrow{q_\alpha} (Y_\alpha,\mu_\alpha)\ : \alpha$ is an ordinal$\rangle$.
Since $\mathtt{Sub}_{\mathsf{M}}(X\times X)$ is a small set, there exists an ordinal β such that $\mathtt{kerp}\, q_\beta = \mathtt{kerp}\, q_{\beta+1}$.
Hence u_β is an isomorphism.
Since $(k_{\beta,1},k_{\beta,2})$ is the pullback of $(y_{\beta,1},y_{\beta,2})$ along q_β, regularity implies v_β is an epimorphism.



Hence $y_{\beta,1}=y_{\beta,2}$, implying $(d_{\beta,1},d_{\beta,2}) = \mathrm{cl}_{\mu_\beta} d_{Y_\beta} \le \mathtt{kerp}\, y_\beta \le d_{Y_\beta}$, i.e., (Y_β,μ_β) is Hausdorff.
Hence $\mathtt{kerp}\, h_X \le \mathtt{kerp}\, q_\beta \le \mathtt{kerp}\, h_X$ implying $(Y_\beta,\mu_\beta) = \left(\mathfrak{h}X, \mu_{\mathfrak{h}}\right)$.

Transfinite construction of Hausdorff reflection

Theorem (see Ghosh, "Internal neighbourhood structures IV: Internal Hausdorff Spaces", Theorem 4.2)

In every reflecting zero context with finite product projections in E and stable regular epimorphisms, there exists a transfinite construction of the Hausdorff reflection of an internal preneighbourhood space (X, μ) *for which* $\mathtt{Sub}_{\mathsf{M}}(X \times X)$ *is a small set.*

This reminds us of a similar proof of the Hausdorff reflection of topological spaces, see, for example Munster, "Hausdorffization and homotopy"; Munster, "The Hausdorff Quotient"

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Thank you. . .