Towards a correspondence theory in region-based theories of space

Presentation · June 2021

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BLAST 2021

Outline

Regions-based theories of space

Boolean contact algebras

Dualities

Points of Boolean contact algebras

Towards a correspondence theory

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Regions-based theories of space

Theories of space that are base on the primitive notions of region (chunk of space), part and some version of nearness like contact or non-tangential part.

The framework of this talk: Boolean contact algebras.

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Boolean contact algebras

A boolean contact algebra is any structure $\langle B, \mathbf{C}, \sqcup, \sqcap, \text{-}, 0, 1\rangle$ such that:

1. $\langle B, \sqcup, \sqcap, \text{-}, 0, 1\rangle$ is a boolean algebra, elements of B are called regions.
2. $\mathbf{C} \subseteq B \times B$ is a contact relation satisfying:

$$0 \not\mathbf{C}\ x \tag{C0}$$

$$x \neq 0 \rightarrow x \ \mathbf{C}\ x \tag{C1}$$

$$x \ \mathbf{C}\ y \rightarrow y \ \mathbf{C}\ x \tag{C2}$$

$$x \ \mathbf{C}\ y \wedge y \leqslant z \rightarrow x \ \mathbf{C}\ z \tag{C3}$$

$$x \ \mathbf{C}\ (y \sqcup z) \rightarrow z \ \mathbf{C}\ y \vee x \ \mathbf{C}\ z\,. \tag{C4}$$

Boolean contact algebras

In a standard way we define three auxiliary relations, overlap, incompatibility and non-tangential part:

$$x \bigcirc y \quad :\longleftrightarrow \quad x \sqcap y \neq 0 \qquad (\mathtt{df}\bigcirc)$$

$$x \perp y \quad :\longleftrightarrow \quad x \sqcap y = 0 \qquad (\mathtt{df}\perp)$$

$$x \ll y \quad :\longleftrightarrow \quad x \not\mathbf{C} -y\,. \qquad (\mathtt{df}\ll)$$

Boolean contact algebras

A canonical interpretation of BCAs is obtained by taking a boolean algebra whose regions are regular open (or regular closed) sets of a topological space, and defining:

$$x \mathbf{C} y \;:\longleftrightarrow\; \mathrm{Cl}\, x \cap \mathrm{Cl}\, y \neq \emptyset\,. \qquad (\mathtt{df}\,\mathbf{C})$$

In consequence:

$$x \ll y \longleftrightarrow \mathrm{Cl}\, x \subseteq y\,.$$

Boolean contact algebras

Another standard interpretation identifies contact with overlap:

$$x \mathbf{C} y \;:\longleftrightarrow\; x \bigcirc y\,, \qquad (\mathtt{df}\,\mathbf{C})$$

In which case:

$$x \ll y \longleftrightarrow x \leqslant y\,.$$

We will call such a contact algebra an overlap algebra.

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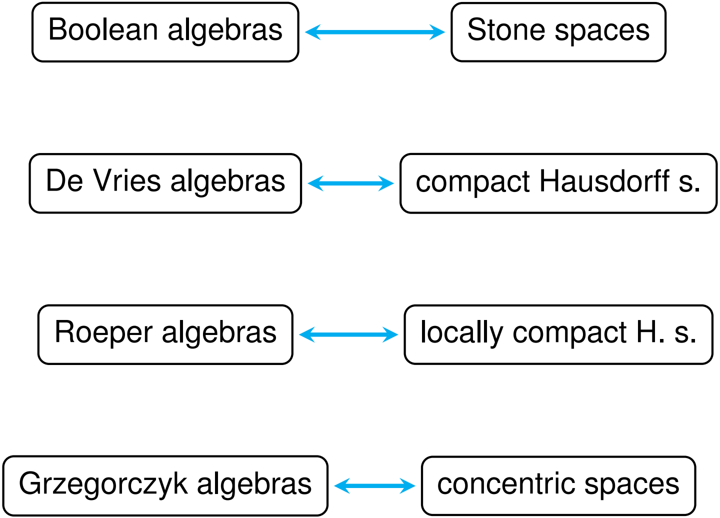
Dualities

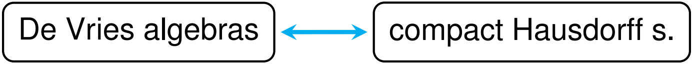

Boolean algebras
Stone spaces
De Vries algebras
compact Hausdorff s.
Roeper algebras
locally compact H. s.
Grzegorczyk algebras
concentric spaces

Dualities

Motivations:

- find concrete representations of Boolean algebras in order to
- understand BAs in an intuitive way.

Dualities



Motivations:

- algebraization of the topological notion of compactness,
- the topological notion was the starting point.

Dualities

Grzegorczyk algebras ⟷ concentric spaces

Motivations:

- spatial intuition about regions and relations between them,
- spatial intuitions about points as sets of «shrinking» regions,
- a characterization of the notion of point was the starting point.

Outline

Regions-based theories of space

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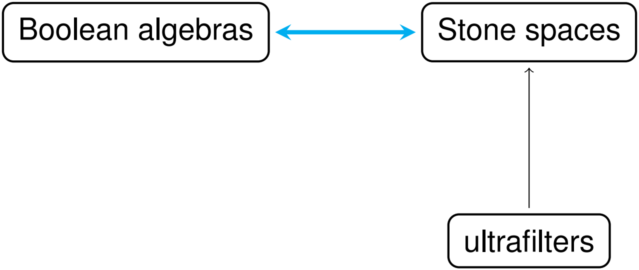
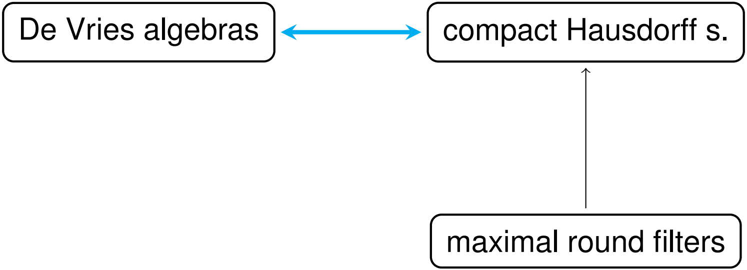
Points



Points


De Vries algebras
compact Hausdorff s.
maximal round filters

Points

$$\downarrow\!\!\!\downarrow x := \{y \in R \mid y \ll x\}\,. \qquad (\mathtt{df}\downarrow\!\!\!\downarrow)$$

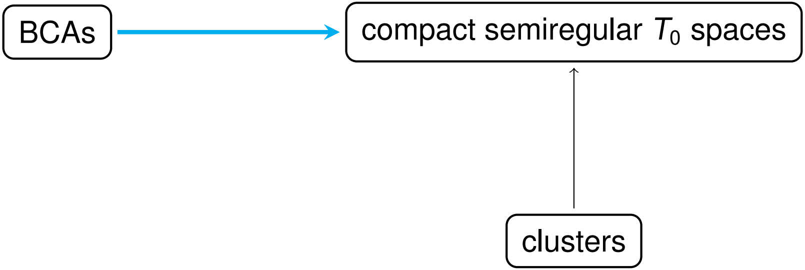
Definition

A proper filter $\mathscr{F}$ of a BCA is round iff for every $x \in \mathscr{F}$: $\mathscr{F} \cap \downarrow\!\!\!\downarrow x \neq \emptyset$.

Definition

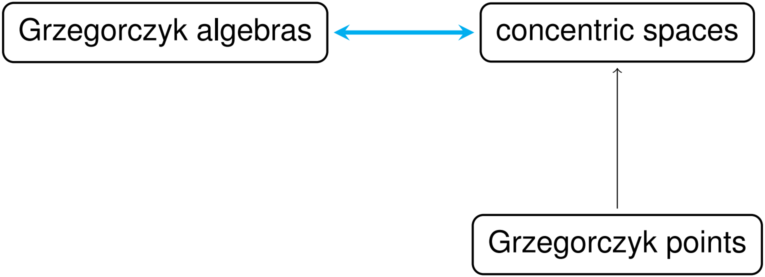
A filter $\mathscr{F}$ is a maximal round filter (a de Vries point) iff $\mathscr{F}$ is maximal in the family of round filters.

Points (representation theorem for BCAs)

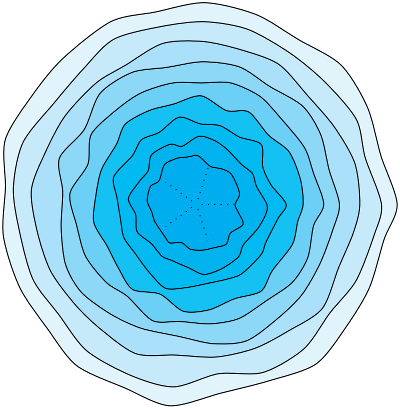


A clan is a non-empty, upward closed set $\mathscr{C}$ of regions such that (a) for $x \sqcup y$ in $\mathscr{C}$, x is in $\mathscr{C}$ or y is in $\mathscr{C}$ and (b) for all $x, y \in \mathscr{C}$, x **C** y. A cluster is a maximal clan.

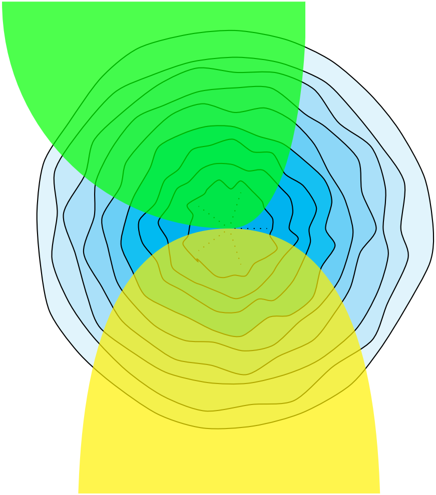
Points


Grzegorczyk algebras
concentric spaces
Grzegorczyk points

Grzegorczyk points



Grzegorczyk points



Grzegorczyk points

A G-representative (or representative of a point), is any nonempty set Q or regions satisfying the following three conditions:

$$0 \notin Q \tag{r0}$$

$$\forall_{x,y \in Q} \left(x = y \vee x \ll y \vee y \ll x \right) \tag{r1}$$

$$\forall_{x \in Q} \exists_{y \in Q} \, y \ll x \tag{r2}$$

$$\forall_{x,y \in R} \big(\forall_{u \in Q} \left(u \bigcirc x \wedge u \bigcirc y \right) \to x \mathbf{\ C}\ y \big) \,. \tag{r3}$$

Let $\mathbf{Q}_G$ be the set of all G-representatives of a given BCA.

Grzegorczyk points

Grzegorczyk points are (proper) filters generated by points representatives:

$$X \in \mathbf{G} \;:\!\longleftrightarrow\; \exists_{Q \in \mathbf{Q}_G} X = \left\{ x \in B \,\middle|\, \exists_{y \in Q}\, y \leqslant x \right\} . \qquad (\mathtt{df}\,\mathbf{G})$$

For a G-point Q, let $\mathscr{F}_Q$ be the G-point generated by Q.

Outline

- Regions-based theories of space
- Boolean contact algebras
- Dualities
- Points of Boolean contact algebras
- Towards a correspondence theory

Correspondences

- Different sets of points of BCAs can be compared with respect to the inclusion relation.
- Statements about those inclusions are formulated in second-order monadic logic.
- Some of those statements are true in BCAs, e.g.:

$$\text{every Grzegorczyk point is a De Vries point}\,.$$

- Some are consequences of or correspond to well-understood properties of BCAs.

Grzegorczyk points and ultrafilters

Theorem

If B is a complete atomless BCA, then no Grzegorczyk point is an ultrafilter.

Proof.

No ultrafilter can be generated by a chain in any complete atomless BA. (Hamkins and Seabold, 2012). □

Grzegorczyk contact algebras

Grzegorczyk contact algebras (Grzegorczyk, 1960) are obtained from BCAs by adding two second-order axioms postulating existence of Grzegorczyk points:

$$\forall_{x \in B} \exists_{Q \in \mathbf{Q}_G} x \in Q \,, \tag{G1}$$

$$x \mathbf{C} y \to \exists_{Q \in \mathbf{Q}_G} \forall_{u \in Q} (u \bigcirc x \wedge u \bigcirc y) \,. \tag{G2}$$

G1 Every region has a point representative.

G2 Point representatives are guaranteed to exists where regions touch each other.

Atoms and Grzegorczyk points

Fact

If a is an atom of a Grzegorczyk contact algebra, then $\{a\}$ is a representative of a point.

Proof sketch.

Since every region is in a point representative, there is Q such that $a \in Q$. But then in Q there is x such that $x \ll a$. So $x = a$ and thus $a \ll a$. It follows that all three conditions for point representatives are satisfied for $\{a\}$. □

Grzegorczyk points and ultrafilters

As we have seen:

Theorem

If B is a complete atomless BCA, then no Grzegorczyk point is an ultrafilter.

We can refine it for GCAs:

Theorem

Any complete Grzegorczyk contact algebra is atomless iff $\mathbf{G} \cap \mathbf{Ult} = \emptyset$.

Proof sketch.

If there is an atom a, then $\{a\}$ is a point representative, and the Grzegorczyk point generated by it is an ultrafilter. □

Grzegorczyk points and ultrafilters

Theorem

In every Grzegorczyk contact algebra, $\mathbf{G} \subseteq \mathbf{Ult}$ *iff every region is isolated:* $x \ll x$.

Proof sketch.

($\rightarrow$) If $x \mathbf{C} -x$, then by (G2) there is a Grzegorczyk point X such that for every $y \in X$, $y \bigcirc x$ and $y \bigcirc -x$. So neither x nor its complement can be in X.

($\leftarrow$) If $X \in \mathbf{G} \setminus \mathbf{Ult}$, then there is a region $y \notin \{0, 1\}$ such that $y \notin X$ and $-y \notin X$. Therefore every $u \in X$ must overlap both y and $-y$, and in consequence $y \mathbf{C} -y$ by properties of G-points. □

Grzegorczyk points and ultrafilters

Theorem

In every complete Grzegorczyk contact algebra the following statements are equivalent:

1. *There are finitely many regions.*
2. *There are finitely many Grzegorczyk points.*
3. $\mathbf{Ult} \subseteq \mathbf{G}$.
4. $\mathbf{Ult} = \mathbf{G}$.

Proof sketch.

Using the classical results that (a) every infinite BA has a free ultrafilter and an infinite antichain and (b) no free filter can be generated by a chain. □

Fréchet filter and Grzegorczyk points

Fréchet filter of an infinite atomic BA (every region is the supremum of a set of atoms) is a set of all regions that miss finitely many atoms.

The sentence "the Fréchet filter is a Grzegorczyk point" is independent from the axioms of Grzegorczyk contact algebras.

Fréchet filter and Grzegorczyk points

Theorem

In no Grzegorczyk contact algebra with uncountably many atoms the Fréchet filter is a Grzegorczyk point.

Proof idea.

By the fact that in no BA with uncountably many atoms the Fréchet filter can be generated by a chain. □

Fact

Every atomic overlap algebra is a Grzegorczyk contact algebra. So there are GCAs in which the Fréchet filter is not a Grzegorczyk point.

Fréchet filter and Grzegorczyk points

In a BA with countably infinitely many atoms define: x **C** y iff x overlaps y or both x and y are composed of infinitely many atoms.

Theorem

Any BA with contact as defined above is a Grzegorczyk contact algebra whose Fréchet filter is a Grzegorczyk point.

Proof idea.

The crucial step is to show that the chain:

$$\{-a_0 \sqcap \ldots \sqcap -a_n \mid n \in \omega\}$$

is a point representative. □

Fréchet filter and Grzegorczyk points

$\mathcal{P}(\omega)$ is an atomic Boolean algebra, that can be turned into a contact algebra via:

$$x \mathbf{C} y \;:\longleftrightarrow\; x \cap y \neq \emptyset \vee \left(|x| = \aleph_0 \wedge |y| = \aleph_0\right) .$$

According to the theorem from the previous slide this is a Grzegorczyk contact algebra whose Fréchet filter is a Grzegorczyk point.

Fréchet filter and Grzegorczyk points

Consider the sentence:

$$\varphi := \text{“Fréchet filter is a G-point”}$$

Theorem

If $\mathfrak{G}$ is an infinite GCA that satisfies φ, then:

1. *$\mathfrak{G}$ has countably many atoms.*
2. *the Fréchet filter is the only G-point that is a free filter.*
3. **G** *has as points (a) all filters generated by atoms and (b) the Fréchet filter (so has countably many points).*

Fréchet filter and Grzegorczyk points

Definition

A topological space is a concetric space iff it is T_0 and every its point has a local basis that satisifes the following condition:

$$U = V \vee \mathrm{Cl}\, U \subseteq V \vee \mathrm{Cl}\, V \subseteq U\,.$$

Theorem (Representation theorem for GCAs)

Every GCA is isomorphic to a dense subalgbera of regular open algebra of a concentric space (whose points are G-points).

Fréchet filter and Grzegorczyk points

$$\varphi := \text{“Fréchet filter is a G-point”}$$

Theorem

Let $\mathfrak{G}$ be an infinite atomic GCA satisfying φ. Then its topological space is a continuous image of the Stone space under the function $f\colon \mathbf{Ult} \to \mathbf{G}$

$$f(U) := \begin{cases} U & \text{if } U \text{ is principal,} \\ \text{the Fréchet filter} & \text{if } U \text{ is free.} \end{cases}$$

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