

Dualities for default bilattices and their applications

Andrew Craig¹ Brian A. Davey² Miroslav Haviar^{3,1}

¹University of Johannesburg, South Africa

²La Trobe University, Melbourne, Australia

³Matej Bel University, Banská Bystrica, Slovakia

BLAST

June 12, 2021

My coauthors



Slovakia, Sept 2017, where we started our joint study of bilattices

Bilattices: a brief introduction

- ▶ Bilattices provide an algebraic tool with which to model simultaneously knowledge and truth.

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- ▶ They were introduced by Belnap in 1977 in a paper entitled **How a computer should think**.
- ▶ Belnap argued that instead of using a logic with two values, for 'true' (***t***) and 'false' (***f***), a computer should use a logic with two further values, for 'contradiction' (**T**) and 'no information' (**\perp**).
- ▶ The resulting structure is equipped with two lattice orders, a **knowledge** or **information order** and a **truth order**, and hence is called a **bilattice**.

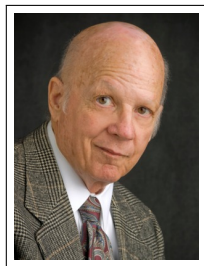
FOUR: Belnap's four-valued logic (1977)

"We want a computer to be able to receive and reason about inconsistent data."

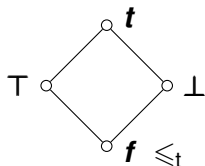
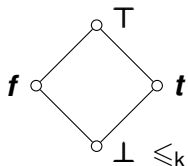
Nuel Belnap

University of Pittsburgh since 1961.

Retired in 2011.



FOUR:



The applications of bilattices

Prioritised default bilattices (in addition to Belnap's *FOUR* they have a hierarchy of default values for 'true' and 'false') have had many applications in **artificial intelligence**.

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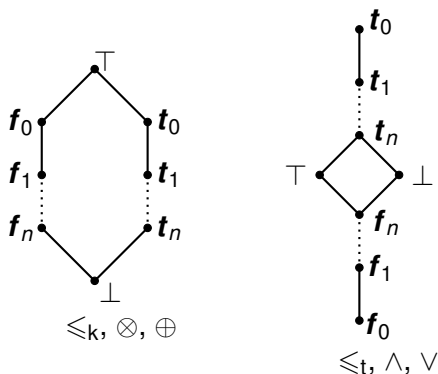
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- ▶ Encheva and Tumin (2007) applied default logic based on a 10-element default bilattice in an **intelligent tutoring system** as a way of resolving problems with contradictory or incomplete input.

Expanding Belnap's four-element bilattice

We studied a new family of default bilattices. The bilattice \mathbf{J}_n drawn in both its knowledge (left) and truth (right) orders:

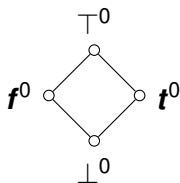


We let $\mathbf{J}_n = \langle J_n; \otimes, \oplus, \wedge, \vee, \neg, C \rangle$ where $C = J_n$ and $\neg t_i = f_i$ and $\neg f_i = t_i$. The negation \neg is an order automorphism w.r.t. the order \leq_k and a dual order automorphism w.r.t. the order \leq_t .

The dualising multi-sorted structure \mathbb{M}_n for

$$\mathcal{V}_n = \text{Var}(\mathbf{J}_n) = \text{ISP}(\{\mathbf{M}_0, \mathbf{M}_1, \dots, \mathbf{M}_n\})$$

- ▶ \leq^j , an order relation on \mathbf{M}_j , for $j \in \{0, \dots, n\}$,



\leq^0

$\top^j \circ$

$\mathbf{1}^j \circ$

$\mathbf{t}^j \circ$

$\mathbf{0}^j \circ$

$\mathbf{f}^j \circ$

$\perp^j \circ$

\leq^j

$\circ \top^k$

$\circ \mathbf{1}^k$

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$\circ \mathbf{f}^k$

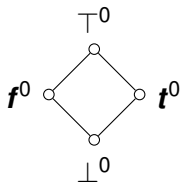
$\circ \perp^k$

\leq^k

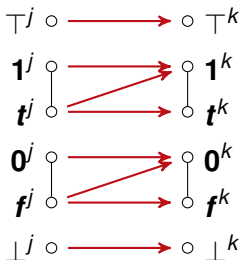
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$$\leq^0$$

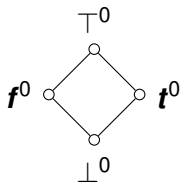


$$\leq^j \quad \leq^{jk} \quad \leq^k$$

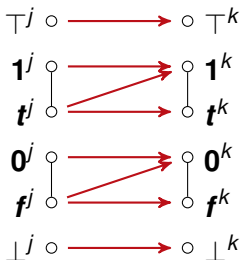
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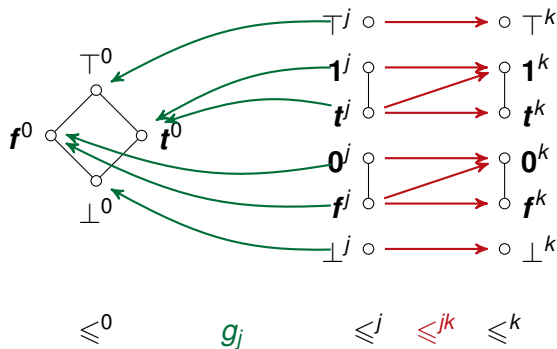


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- ▶ $g_j: \mathbf{M}_j \rightarrow \mathbf{M}_0$, the operation that maps $\{f^j, \mathbf{0}^j\}$ to f^0 and maps $\{t^j, \mathbf{1}^j\}$ to t^0 , for $j \in \{1, \dots, n\}$.



A natural duality for the variety $\mathcal{V}_n = \text{Var}(\mathbf{J}_n)$

Theorem

Let $n \in \omega \setminus \{0\}$. Define the multi-sorted alter ego

$$\mathbb{M}_n = \langle \mathbf{M}_0 \dot{\cup} \mathbf{M}_1 \dot{\cup} \dots \dot{\cup} \mathbf{M}_n; \mathcal{G}_{(n)}, \mathcal{S}_{(n)}, \mathcal{T} \rangle,$$

where

$$\mathcal{G}_{(n)} = \{ \mathbf{g}_k \mid k \in \{1, \dots, n\} \}, \text{ and}$$

$$\mathcal{S}_{(n)} = \{ \leq^k \mid k \in \{0, \dots, n\} \} \cup \{ \leq^{jk} \mid j, k \in \{1, \dots, n\} \text{ with } j < k \}.$$

The alter ego \mathbb{M}_n yields a duality that is both optimal and full on the variety $\mathcal{V}_n = \text{Var}(\mathbf{J}_n) = \text{ISP}(\{\mathbf{M}_0, \mathbf{M}_1, \dots, \mathbf{M}_n\})$.

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Our first aim is to give a description of the dual category

$$\mathfrak{X}_n := \text{IS}_c \mathbf{P}^+(\mathbb{M}_n).$$

Theorem

$\mathbb{X} = \langle X_0 \dot{\cup} \dots \dot{\cup} X_n; \mathcal{G}_{(n)}, \mathcal{S}_{(n)}, \mathcal{T} \rangle$ belongs to $\mathcal{X}_n := \text{IS}_c\text{P}^+(\mathbb{M}_n)$ iff

(A1) $g_k: X_k \rightarrow X_0$ is continuous, for all $k \in [1, n]$;

(A2) $(\forall k \in [1, n]) (\forall x, y \in X_k) x \leq^k y \implies g_k(x) = g_k(y)$;

(A3) $(\forall j, k \in [1, n] \text{ with } j < k)$

$$(\forall x \in X_j)(\forall y \in X_k) x \leq^{jk} y \implies g_j(x) = g_k(y);$$

(A4) $(\forall j, k \in [1, n] \text{ with } j < k) (\forall x, y \in X_j)(\forall u, v \in X_k)$

$$x \leq^j y \ \& \ y \leq^{jk} u \ \& \ u \leq^k v \implies x \leq^{jk} z;$$

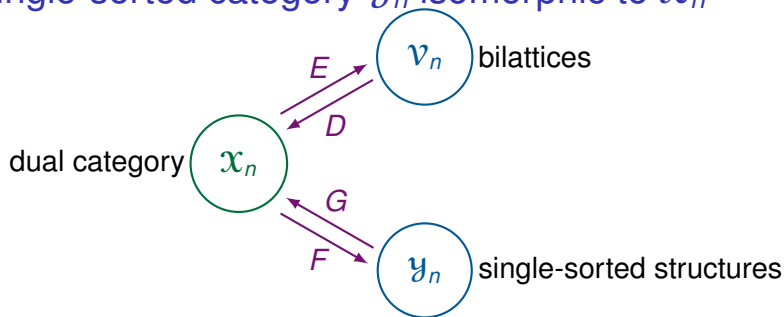
(A5) $(\forall j, k, \ell \in [1, n] \text{ with } j < k < \ell) (\forall x \in X_j)(\forall y \in X_k)(\forall z \in X_\ell)$

$$x \leq^{jk} y \ \& \ y \leq^{k\ell} z \implies x \leq^{j\ell} z;$$

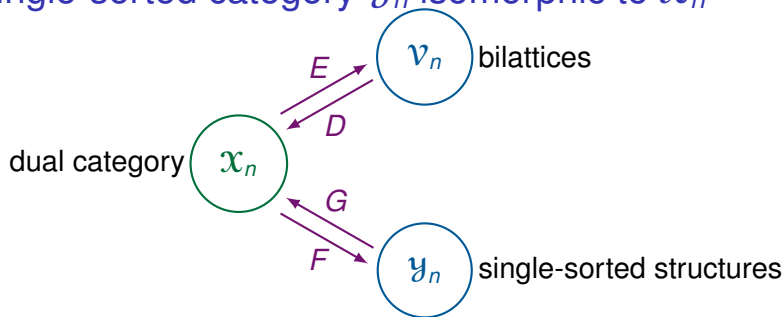
(A6) $(\forall k \in [0, n]) \langle X_k; \leq^k, \mathcal{T}_k \rangle$ is a Priestley space ($\mathcal{T}_k := \mathcal{T}|_{X_k}$);

(A7) $(\forall j, k \in [1, n] \text{ with } j < k) (\forall x \in X_j)(\forall y \in X_k)$ with $x \not\leq^{jk} y$, there exist U_j, U_{j+1}, \dots, U_k , with U_ℓ a clopen up-set of $\langle X_\ell; \leq^\ell, \mathcal{T}_\ell \rangle$, for all $\ell \in [j, k]$, such that U_j, \dots, U_k are mutually increasing with $x \in U_j$ and $y \in X_k \setminus U_k$.

A single-sorted category \mathcal{Y}_n isomorphic to \mathcal{X}_n



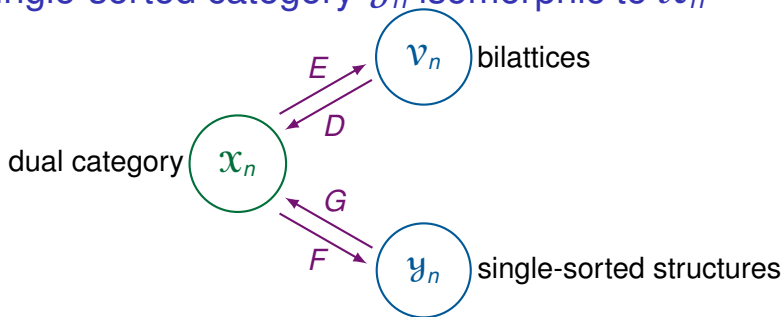
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For $\mathbb{X} = \langle X; \mathcal{G}_{(n)}, \mathcal{S}_{(n)}, \mathcal{T} \rangle \in \mathcal{X}_n$ where $X = X_0 \dot{\cup} \dots \dot{\cup} X_n$ we define a binary relation \preceq on X as follows:

$$x \preceq y :\Leftrightarrow \begin{cases} (\exists k \in [0, n]) x, y \in X_k \text{ and } x \leq^k y, \text{ or} \\ (\exists j, k \in [1, n] \text{ with } j < k) x \in X_j \ \& \ y \in X_k \ \& \ x \leq^{jk} y. \end{cases}$$

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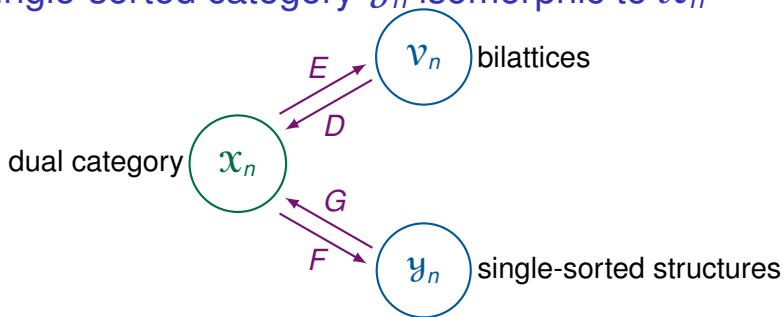


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(a) \mathbb{X} satisfies (A4)–(A6) iff \preceq is an order on X .

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- (a) \mathbb{X} satisfies (A4)–(A6) iff \preceq is an order on X .
- (b) \mathbb{X} satisfies (A4)–(A7) iff $\langle X; \preceq, \mathcal{T} \rangle$ is a Priestley space.

A single-sorted category \mathcal{Y}_n isomorphic to \mathcal{X}_n

Given $n \in \omega \setminus \{0\}$, define an **n -ranking** of any Priestley space $\langle X; \preceq, \mathcal{T} \rangle$ to be a continuous order-preserving map, rnk , from $\langle X; \preceq, \mathcal{T} \rangle$ to the finite Priestley space $\langle [0, n]; \leq, \mathcal{T} \rangle$, where \leq is the usual order inherited from \mathbb{Z} and \mathcal{T} is the discrete topology.

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Given $\mathbb{X} \in \mathcal{X}_n$, we can enrich the Priestley space $\langle X; \preceq, \mathcal{T} \rangle$ with a unary operation g and an n -ranking function defined as follows:

- ▶ $g: X \rightarrow X$ is defined by $g(x) := x$ for all $x \in X_0$ and $g(x) := g_k(x)$, for all $x \in X_k$ and all $k \in [1, n]$,

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We will define a functor $F: \mathcal{X}_n \rightarrow \mathcal{Y}_n$ by

$$F(\mathbb{X}) := \langle X; \preceq, g, \text{rnk}, \mathcal{T} \rangle.$$

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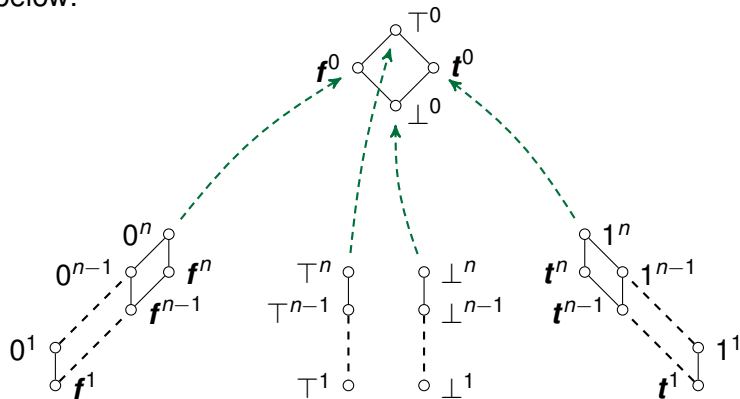
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Before **abstractly** defining \mathcal{Y}_n , we present an example of $F(\mathbb{X})$.

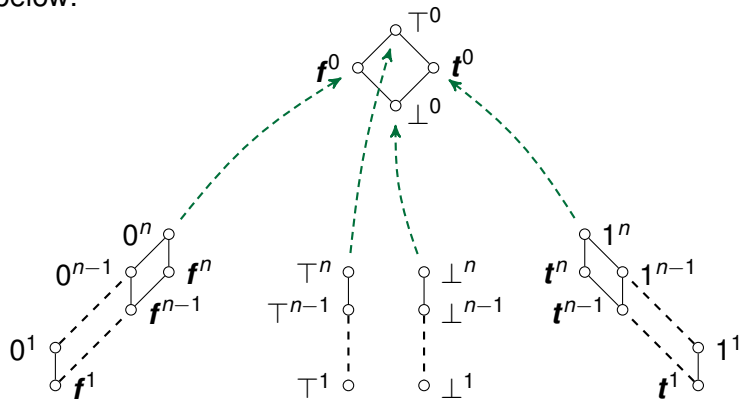
A single-sorted category \mathcal{Y}_n isomorphic to \mathcal{X}_n

$F(\mathbb{M}_n) = \langle M_0 \dot{\cup} M_1 \dot{\cup} \dots \dot{\cup} M_n; \preceq, g, \text{rnk}, \mathcal{T} \rangle = F(D(\mathbf{F}_{\mathcal{V}_n}(1)))$ is below:



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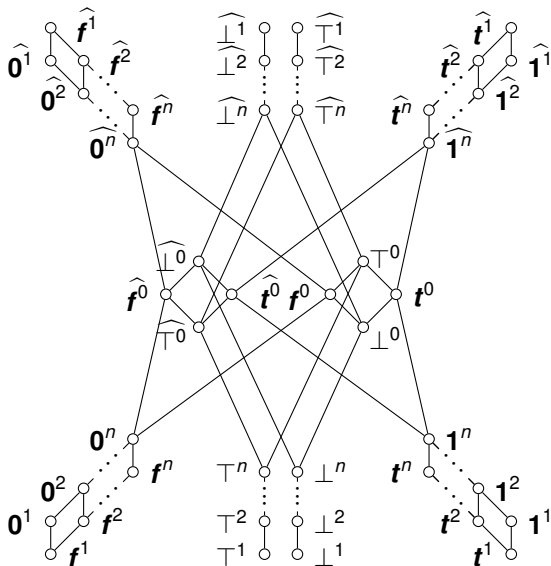
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It is interesting to compare this structure with the Priestley dual

$$H(\mathbf{F}_{V_n}(1)^b).$$

The Priestley dual of $\mathbf{F}_{V_n}(1)^b$



A single-sorted category \mathcal{Y}_n isomorphic to \mathcal{X}_n

We **abstractly** define \mathcal{Y}_n to be the category whose objects are topological structures $\langle X; \preceq, g, \text{rnk}, \mathcal{T} \rangle$ satisfying (B1)–(B6) below and whose morphisms are continuous maps preserving \preceq , g and rnk .

- (B1) $\langle X; \preceq, \mathcal{T} \rangle$ is a Priestley space,
- (B2) g is a continuous retraction,
- (B3) $x \preceq y$ implies $g(x) = g(y)$, for all $x, y \in X$,
- (B4) $g(X)$ is a union of order components of $\langle X; \preceq \rangle$,
- (B5) $\text{rnk}: X \rightarrow [0, n]$ is an n -ranking of $\langle X; \preceq, \mathcal{T} \rangle$,
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(B6) $g(X) = \{ x \in X \mid \text{rnk}(x) = 0 \}$.

Note: The topological structures in \mathcal{Y}_n can be made first order by removing the ‘operation’ rnk , adding $n + 1$ topologically closed unary relations X_0, \dots, X_n , and replacing the assumption that rnk is order-preserving by the assumption that the X_k partition X plus the following axioms:

$$\begin{aligned} (\forall k \in [0, n]) \quad (\forall x, y \in X) \quad x \in X_k \ \& \ x \preceq y \\ \implies y \in X_k \ \text{or} \ y \in X_{k+1} \ \text{or} \ \dots \ \text{or} \ y \in X_n. \end{aligned}$$

A single-sorted category \mathcal{Y}_n isomorphic to \mathcal{X}_n

Let $\mathbb{Y} = \langle Y; \preceq, g, \text{rnk}, \mathcal{T} \rangle$ be an object in \mathcal{Y}_n . We define

$$G(\mathbb{Y}) := \langle X_0 \dot{\cup} \dots \dot{\cup} X_n; \mathcal{G}_{(n)}, \mathcal{S}_{(n)}, \mathcal{T} \rangle$$

in the signature of \mathbb{M}_n as follows:

- ▶ $X_k := \{x \in Y \mid \text{rnk}(x) = k\}$, for all $k \in [0, n]$,
- ▶ $g_k: X_k \rightarrow X_0$ is given by $g_k := g \upharpoonright_{X_k}$, for all $k \in [1, n]$,
- ▶ $\preceq^k := \preceq \cap (X_k \times X_k)$, for all $k \in [0, n]$,
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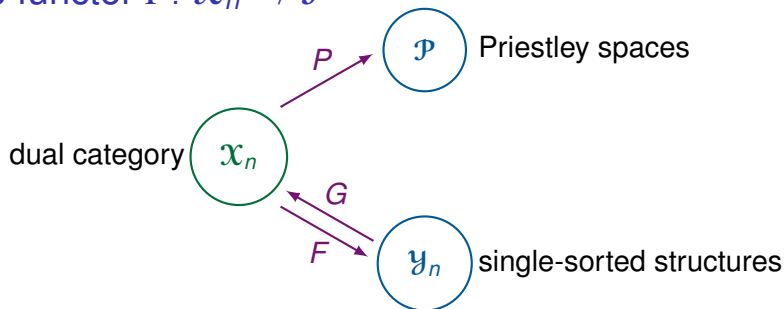
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Theorem

$F: \mathcal{X}_n \rightarrow \mathcal{Y}_n$ and $G: \mathcal{Y}_n \rightarrow \mathcal{X}_n$ are well-defined, mutually inverse category isomorphisms. In particular,

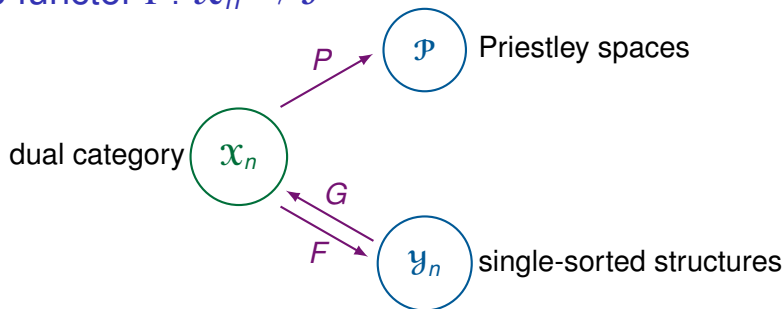
- ▶ $G(F(\mathbb{X})) = \mathbb{X}$, for all $\mathbb{X} \in \mathcal{X}_n$, and
- ▶ $F(G(\mathbb{Y})) = \mathbb{Y}$, for all $\mathbb{Y} \in \mathcal{Y}_n$.

The functor $P: \mathcal{X}_n \rightarrow \mathcal{P}$



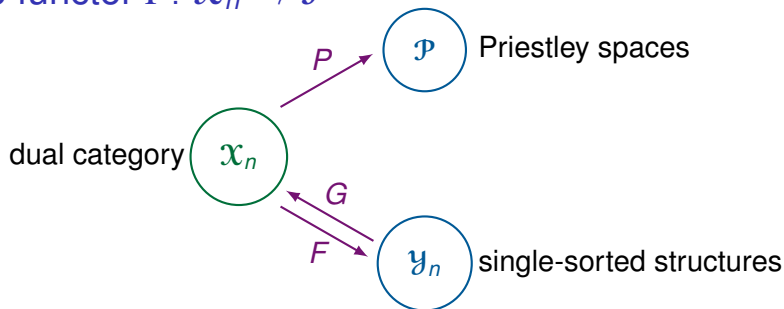
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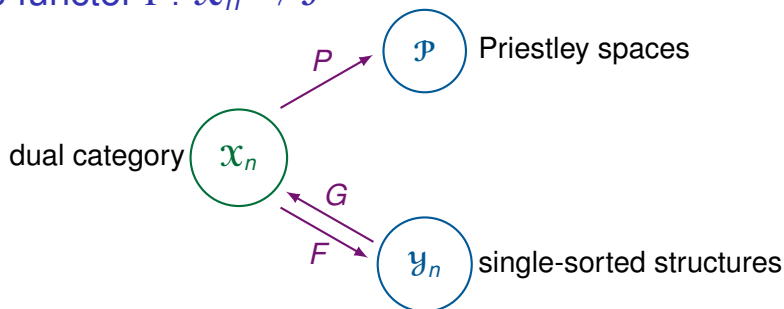
- ▶ We now describe $P: \mathcal{X}_n \rightarrow \mathcal{P}$ (Priestley spaces).
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Definition

Let $\mathbb{X} = \langle X_0 \dot{\cup} \dots \dot{\cup} X_n; \mathcal{G}_{(n)}, \mathcal{S}_{(n)}, \mathcal{T} \rangle$ be an object in \mathcal{X}_n , define $X := X_0 \dot{\cup} \dots \dot{\cup} X_n$ and let $F(\mathbb{X}) = \langle X; g, \leq, \text{rnk}, \mathcal{T} \rangle$ be the corresponding object in \mathcal{Y}_n . We define $P(\mathbb{X}) := \langle X \dot{\cup} \widehat{X}; \preceq, \mathcal{T} \rangle$ where $\widehat{X} := \{\widehat{x} \mid x \in X\}$, \mathcal{T} is the disjoint union topology, and the order \preceq is given on $X \dot{\cup} \widehat{X}$ as on the next page.

The order in the constructed Priestley spaces

Definition

- ▶ for $x, y \in X$: $x \preceq y \iff x \leq y$,
- ▶ for $\hat{x}, \hat{y} \in \hat{X}$: $\hat{x} \preceq \hat{y} \iff x \geq y$,
- ▶ for $x \in X \setminus X_0$ and $y \in X_0$: $x \preceq y \iff g(x) \leq y$,
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The last case of the definition of \preceq can be thought of as obtaining $x \preceq \hat{y}$ by passing through X_0 (via $g(x) \leq g(y)$) or through \hat{X}_0 (via $g(x) \geq g(y)$).

Example of $P(\mathbb{X})$

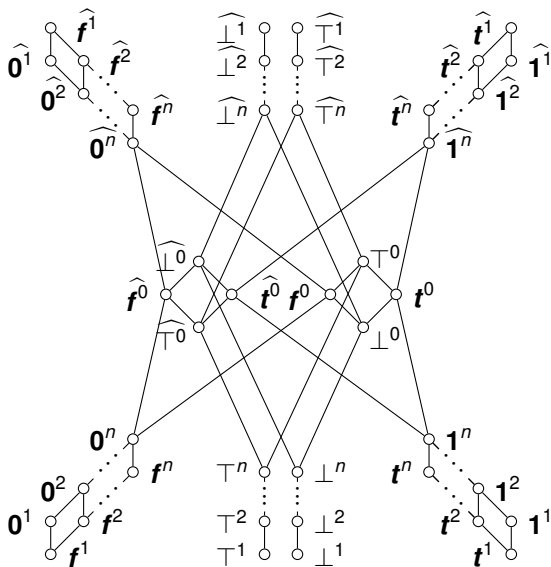


Figure: The ordered set $P(\mathbb{M}_n)$ - the Priestley dual of $\mathbf{F}_{v_n}(1)^b$.

Theorem and its application

Theorem

Fix $n \in \omega \setminus \{0\}$. Then $\mathcal{P}: \mathcal{X}_n \rightarrow \mathcal{P}$ is a well-defined functor. For each prioritised default bilattice $\mathbf{A} \in \mathcal{V}_n$, let $\mathbf{A}^b = \langle \mathbf{A}; \wedge, \vee, \mathbf{f}_0, \mathbf{t}_0 \rangle$ be its bounded-distributive lattice truth reduct. Then its Priestley dual $H(\mathbf{A}^b)$ is isomorphic to $\mathcal{P}(D(\mathbf{A}))$.

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- ▶ We counted the down-sets of $P(\mathbb{M}_n)$ by first counting the number that do not intersect the top \mathbf{T} and then counting the number that do.

The calculation

Claim

The number of down-sets of $\mathbb{P}(\mathbb{M}_n)$ that do not intersect \mathbf{T} is

$$f(n) = \frac{1}{4}(n^6 + 10n^5 + 41n^4 + 96n^3 + 148n^2 + 148n + 144).$$

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Let U be a down-set of $P(\mathbb{M}_n)$ that does not intersect \mathbf{T} . The intersection $U \cap \mathbf{C}$ is one of the 36 down-sets of \mathbf{C} . The number of such U for a given intersection $U \cap \mathbf{C}$ is given in the table:

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$U \cap \mathbf{C}$	# of such U
\emptyset	$\frac{1}{4}(n+1)^4(n+2)^2$
$\{\perp\}$ or $\{\widehat{\mathbf{T}}\}$	$\frac{1}{4}(n+1)^3(n+2)^2$
$\{\perp, \mathbf{f}\}$ or $\{\perp, \mathbf{t}\}$ or $\{\widehat{\mathbf{T}}, \widehat{\mathbf{f}}\}$ or $\{\widehat{\mathbf{T}}, \widehat{\mathbf{t}}\}$	$\frac{1}{2}(n+1)^2(n+2)$
$\{\perp, \mathbf{f}, \mathbf{t}\}$ or $\{\widehat{\mathbf{T}}, \widehat{\mathbf{f}}, \widehat{\mathbf{t}}\}$	$n+1$
$\{\perp, \widehat{\mathbf{T}}\}$	$\frac{1}{4}(n+1)^2(n+2)^2$
$\{\perp, \widehat{\mathbf{T}}, \widehat{\mathbf{f}}\}$ or $\{\perp, \widehat{\mathbf{T}}, \widehat{\mathbf{t}}\}$ or $\{\perp, \mathbf{f}, \widehat{\mathbf{T}}\}$ or $\{\perp, \mathbf{t}, \widehat{\mathbf{T}}\}$ or $\{\perp, \mathbf{f}, \widehat{\mathbf{T}}, \widehat{\mathbf{f}}\}$ or $\{\perp, \mathbf{t}, \widehat{\mathbf{T}}, \widehat{\mathbf{t}}\}$	$\frac{1}{2}(n+1)(n+2)$
each of the remaining 20 possibilities	1

The calculation - continuation

Claim

The number of down-sets of $P(\mathbb{M}_n)$ that intersect the top \mathbf{T} is

$$g(n) = \frac{1}{4}(n^6 + 10n^5 + 43n^4 + 108n^3 + 166n^2 + 148n).$$

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The number of down-sets of $P(\mathbb{M}_n)$ that intersect the top \mathbf{T} is

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A down-set U of $P(\mathbb{M}_n)$ that intersects \mathbf{T} does intersect the set $\min(\mathbf{T}) = \{\widehat{\mathbf{0}}^n, \widehat{\perp}^n, \widehat{\top}^n, \widehat{\mathbf{1}}^n\}$ of minimal elements of \mathbf{T} in one of the 15 non-empty subsets of $\min(\mathbf{T})$ given in the table below:

$U \cap \min(\mathbf{T})$	# of such U
$\{\widehat{\mathbf{0}}\}$ or $\{\widehat{\mathbf{1}}\}$	$(\frac{1}{2}(n+1)(n+2) - 1)(\frac{1}{2}(n+1)(n+2) + 8)$
$\{\widehat{\perp}\}$ or $\{\widehat{\top}\}$	$5n$
$\{\widehat{\mathbf{0}}, \widehat{\mathbf{1}}\}$	$4(\frac{1}{2}(n+1)(n+2) - 1)^2$
$\{\widehat{\mathbf{0}}, \widehat{\top}\}$ or $\{\widehat{\mathbf{0}}, \widehat{\perp}\}$ or $\{\widehat{\top}, \widehat{\mathbf{1}}\}$ or $\{\widehat{\perp}, \widehat{\mathbf{1}}\}$	$3n(\frac{1}{2}(n+1)(n+2) - 1)$
$\{\widehat{\top}, \widehat{\perp}\}$	n^2
$\{\widehat{\perp}, \widehat{\top}, \widehat{\mathbf{1}}\}$ or $\{\widehat{\mathbf{0}}, \widehat{\top}, \widehat{\perp}\}$	$n^2(\frac{1}{2}(n+1)(n+2) - 1)$
$\{\widehat{\mathbf{0}}, \widehat{\top}, \widehat{\mathbf{1}}\}$ or $\{\widehat{\mathbf{0}}, \widehat{\perp}, \widehat{\mathbf{1}}\}$	$2n(\frac{1}{2}(n+1)(n+2) - 1)^2$
$\{\widehat{\mathbf{0}}, \widehat{\perp}, \widehat{\top}, \widehat{\mathbf{1}}\}$	$n^2(\frac{1}{2}(n+1)(n+2) - 1)^2$

The final result

Theorem

Let $n \in \omega \setminus \{0\}$. Then the cardinality of the 1-generated free algebra in the variety $\mathcal{V}_n = \text{Var}(\mathbf{J}_n)$ is

$$|F_{\mathcal{V}_n}(1)| = \frac{1}{2}(n^6 + 10n^5 + 42n^4 + 102n^3 + 157n^2 + 148n + 72).$$



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Thank you for your attention!