Katětov order

Michael Hrušák

CCM
Universidad Nacional Autónoma de México
michael@matmor.unam.mx

Morelia/Las Cruces
2021

Notation: Ideals, filters and ultrafilters

- $\mathcal{I} \subsetneq \mathcal{P}(\omega)$ is an ideal if (1) it is hereditary, (2) closed under finite unions, (3) contains all finite sets, (4) $\omega \notin \mathcal{I}$.
- $\mathcal{F} \subsetneq \mathcal{P}(\omega)$ is an filter if (1) it is closed under supersets, (2) closed under finite intersections, (3) contains all co-finite sets, (4) $\emptyset \notin \mathcal{F}$.
- $\mathcal{U} \subsetneq \mathcal{P}(\omega)$ is an ultrafilter if it is a maximal (free) filter.
- An ideal $\mathcal{I}$ is tall if every infinite subset of ω contains an infinite set in $\mathcal{I}$.
- Given and ideal $\mathcal{I}$, $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ denotes the family of $\mathcal{I}$-positive sets.
- Given $\mathcal{X} \subseteq \mathcal{P}(\omega)$, $\mathcal{X}^* = \{\omega \setminus X : X \in \mathcal{X}\}$.
- Given $\mathcal{X} \subseteq \mathcal{P}(\omega)$ and $Y \in \mathcal{P}(\omega)$,
  $\mathcal{X} \restriction Y = \{X \cap Y : X \in \mathcal{X}\}$ or $\{X \cap Y : X \in \mathcal{X}$ s.t. $|X \cap Y| = \omega\}$.

Notation: Almost disjoint families

- An infinite $\mathcal{A} \subsetneq [\omega]^\omega$ is almost disjoint (AD) if $A \cap B$ is finite for any $A \neq B \in \mathcal{A}$
- Given an AD family $\mathcal{A}$,
  - $\mathcal{I}(\mathcal{A}) = \{X \subseteq \omega : \exists \mathcal{B} \in [\mathcal{A}]^{<\omega}\ X \subseteq^* \bigcup \mathcal{B}\}$
  - $\mathcal{I}^+(\mathcal{A}) = \mathcal{I}(\mathcal{A})^+ = \mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A})$, while
- An AD family $\mathcal{A}$ is MAD if for every $X \in [\omega]^\omega$ there is an $A \in \mathcal{A}$ such that $A \cap X$ is infinite (iff $\mathcal{I}(\mathcal{A})$ is tall).

Katětov order

Definition (Katětov '68).

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω.

- (Katětov order) $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that $f^{-1}[I] \in \mathcal{J}$, for all $I \in \mathcal{I}$.
- (Katětov-Blass order) As above with a finite-to-one function f.
- We will say $\mathcal{I}$ and $\mathcal{J}$ are Katětov-equivalent ($\mathcal{I} \simeq_K \mathcal{J}$) if $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \leq_K \mathcal{I}$, and analogously for the Katětov-Blass order.

The basic properties of the Katětov order are listed here.
Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω.

1. $\mathcal{I} \simeq_K \text{fin}$ if and only if $\mathcal{I}$ is not tall.
2. If $\mathcal{I} \subseteq \mathcal{J}$ then $\mathcal{I} \leq_K \mathcal{J}$.
3. If $X \in \mathcal{I}^+$ then $\mathcal{I} \leq_K \mathcal{I} \restriction X$.
4. $\mathcal{I} \oplus \mathcal{J} \leq_K \mathcal{I}, \mathcal{J}$.
5. $\mathcal{I}, \mathcal{J} \leq_K \mathcal{I} \times \mathcal{J}$.

Here $\mathcal{I} \oplus \mathcal{J}$ denotes the disjoint sum of $\mathcal{I}$ and $\mathcal{J}$, and

$$\mathcal{I} \times \mathcal{J} = \{A \subseteq \omega \times \omega : \{n : \{m : (n, m) \in A\} \notin \mathcal{J}\} \in \mathcal{I}\}$$

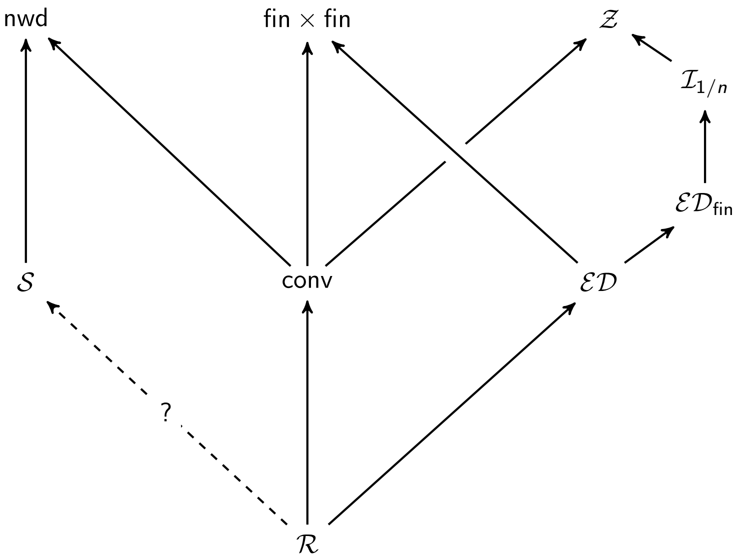
denotes the *Fubini product* of the ideals.

It is easy to see that both the disjoint sum and the Fubini product of Borel ideals are Borel. Hence, the Katětov order on Borel ideals is both upward and downward directed.

Up and down

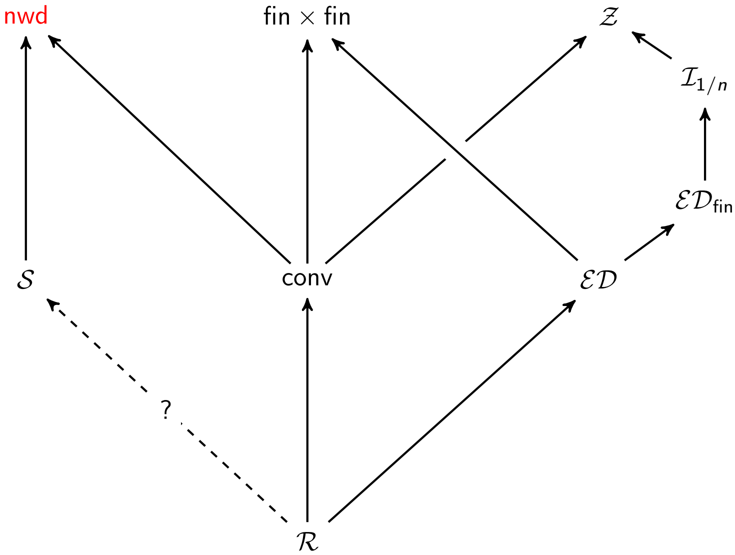
1. Maximal ideals (ultrafilters) are cofinal in the Katětov order on (t)all ideals
2. (Ideals generated by) MAD families are co-initial in the Katětov order on tall ideals
3. (Sakai '18) Every analytic ideal is contained in a Borel ideal.
4. (Grebík-Vidnyánszky '21) Every tall analytic ideal contains a tall F_σ-ideal.
5. Borel ideals are both co-final and co-initial among analytic ideals in the Katětov order.

The diagram

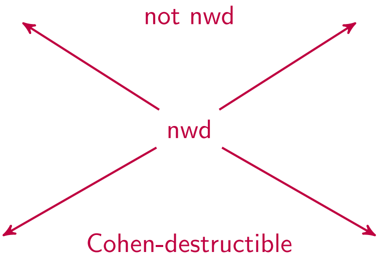


nwd
fin × fin
$\mathcal{Z}$
$\mathcal{I}_{1/n}$
$\mathcal{ED}_{\mathrm{fin}}$
$\mathcal{S}$
conv
$\mathcal{ED}$
?
$\mathcal{R}$

The ideal *nwd*

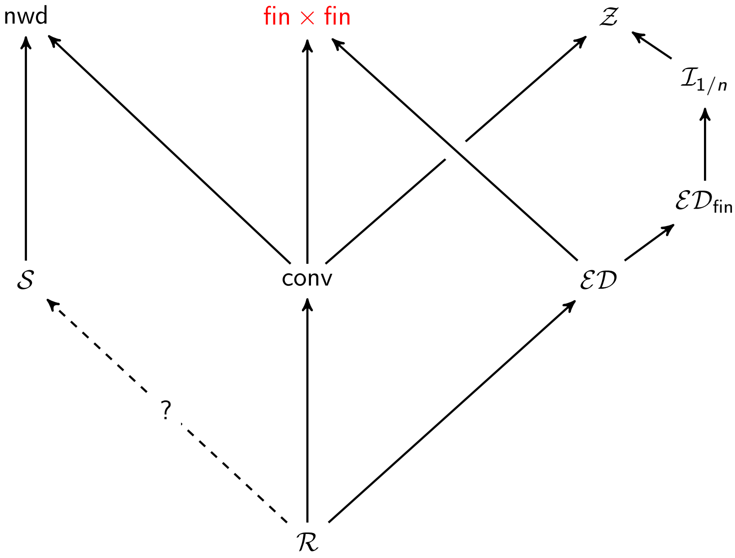


Upward and downward cones of Borel ideals



- (Baumgartner '95) An ultrafilter $\mathcal{U}$ is an nwd-ultrafilter if for every $f : \omega \to \mathbb{Q}$ there is a $U \in \mathcal{U}$ such that $f[U] \in nwd$, i.e. $nwd \not\leq_K \mathcal{U}^*$.
- (Błaszczyk-Shelah '01) There is a σ-centered forcing not adding Cohen-reals iff there is a *nwd*-ultrafilter.
- (Shelah '98) It is consistent that there are no *nwd*-ultrafilters.
- (Prikry '70s) Is it consistent that every c.c.c forcing adds a Cohen or random real?
- (H., Kurilić '01) A MAD family $\mathcal{A}$ is Cohen-indestructible iff $\mathcal{I}(\mathcal{A}) \not\leq_K nwd$.
- (Steprāns '93) Is there a Cohen-indestructible MAD family?

The ideal *fin × fin*



The ideal $\mathit{fin} \times \mathit{fin}$

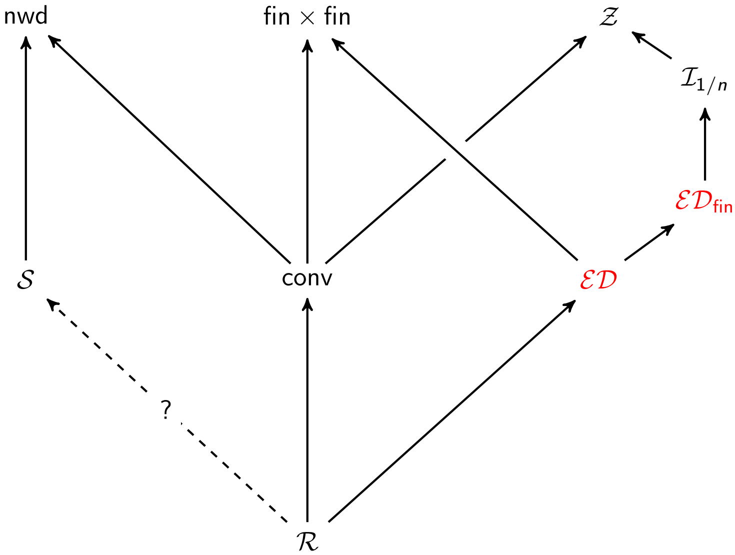
$\mathit{fin} \times \mathit{fin} = \{A \subseteq \omega\times\omega : \{n \in \omega : \{m \in \omega : (n, m) \in A\} \text{ is infinite}\} \text{ is finite}\}$

- An ultrafilter $\mathcal{U}$ is a P-point iff $\mathit{fin} \times \mathit{fin} \not\leq_K \mathcal{U}^*$
- $\mathcal{I}(\mathcal{A}) \leq_K \mathit{fin} \times \mathit{fin}$ for every MAD family $\mathcal{A}$,
- (Laczkovich-Recław '09) For Borel ideal $\mathcal{I}$, $\mathit{fin} \times \mathit{fin} \not\leq_K \mathcal{I}$ iff $\mathcal{I}$ and $\mathcal{I}^*$ can be separated (in $\mathcal{P}(\omega)$) by an F_σ-set.
- (Solecki '00) Every $F_{\sigma\delta}$-ideal can be separated from its dual by an F_σ-set.
- No $F_{\sigma\delta}$-ideal is Katětov above $\mathit{fin} \times \mathit{fin}$.
- (Kwela '21) There is an ideal $\mathcal{I}$ which cannot be extended to an $F_{\sigma\delta}$-ideal, yet $\mathit{fin} \times \mathit{fin} \not\leq_K \mathcal{I}$.

Problem

Let $\mathcal{I}$ be a Borel ideal. Can $\mathcal{I}$ be extended to an $F_{\sigma\delta}$-ideal or $\mathit{fin} \times \mathit{fin} \not\leq_K \mathcal{I} \restriction X$ for some $X \in \mathcal{I}^+$?

The eventually different ideals



The eventually different ideals

$$\mathcal{ED} = \{A \subseteq \omega \times \omega : \ \exists n \in \omega \ \forall m \geq n \ |\{k \in \omega : (m,k) \in A\}| < n\}$$

and

$$\mathcal{ED}_{fin} = \mathcal{ED} \restriction \Delta \text{ where } \Delta = \{(m,n) : n \leq m\}.$$

- $\mathcal{ED} \not\leq_K \mathcal{I}$ iff every partition of ω into sets in $\mathcal{I}$ has a positive selector.
- $\mathcal{ED}_{fin} \not\leq_{KB} \mathcal{I}$ iff every partition of ω into finite sets has a positive selector.
- For a Borel ideal $\mathcal{I}$, $\mathcal{ED}_{fin} \leq_{KB} \mathcal{I}$ iff $\mathcal{I}$ is ω-hitting i.e. for every sequence $\{A_n : n \in \omega\}$ of infinite subsets of ω there is an $I \in \mathcal{I}$ such that $|A_n \cap I| = \omega$ for all $n \in \omega$.
- $\forall X \in \mathcal{I}^+ \ \mathcal{ED} \not\leq_K \mathcal{I} \restriction X$ iff $\mathcal{I}^+ \to (\omega, I^+)^2_2$.

The Category dichotomy

Theorem (H. '17)

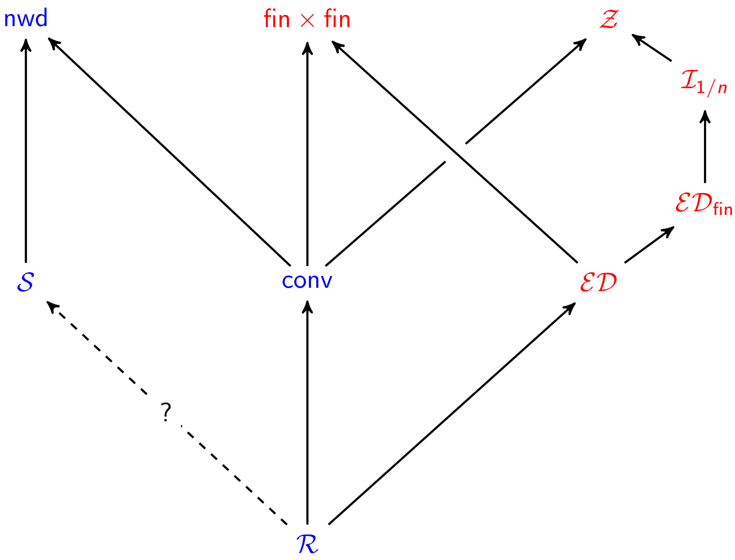
Given a Borel ideal $\mathcal{I}$*, either* $\mathcal{I} \leq_K \mathsf{nwd}$ *or* $\exists X \in \mathcal{I}^+\ \mathcal{ED} \leq_K \mathcal{I} \restriction X$.

Corollary (H.)

Given a Borel ideal $\mathcal{I}$*, either* $\mathcal{I} \leq_K \mathsf{nwd}$ *or* $\exists X \in \mathcal{I}^+\ \mathsf{fin} \times \mathsf{fin} \leq_K \mathcal{I} \restriction X$ *or* $\exists X \in \mathcal{I}^+\ \mathcal{ED}_{\mathsf{fin}} \leq_{KB} \mathcal{I} \restriction X$.

Problem

Is there (in ZFC) an ideal not satisfying the Category dichotomy?


nwd
fin × fin
$\mathcal{Z}$
$\mathcal{I}_{1/n}$
$\mathcal{ED}_{\mathrm{fin}}$
$\mathcal{S}$
conv
$\mathcal{ED}$
?
$\mathcal{R}$

No minimal tall Borel ideal

Theorem (Grebik-H. '20)

There is no tall analytic ideal $\mathcal{J}$ Katětov below all tall F_σ-ideals.

Corollary (Grebik-H. '20)

There is no Katětov-minimal tall Borel ideal.

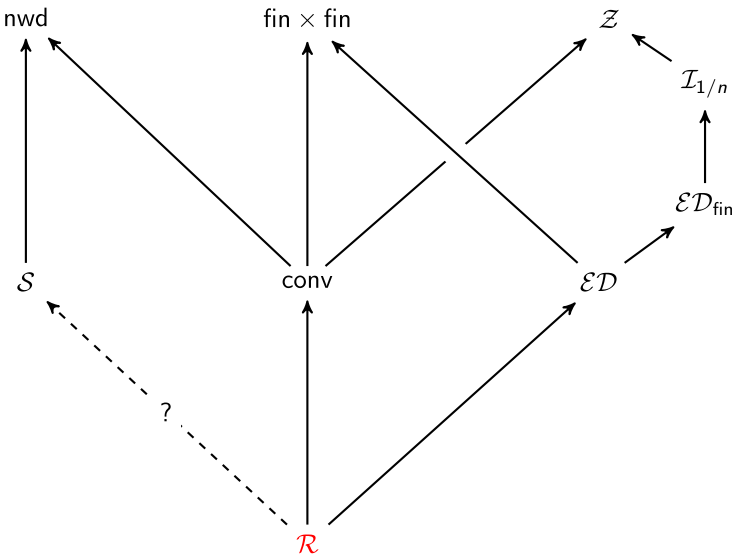
The proof is "non-constructive".

Problem (Uzcategui)

Is there for every tall Borel ideal a Borel function $\varphi : [\omega]^\omega \to \mathcal{I} \cap [\omega]^\omega$ such that $\varphi(A) \subseteq A$ for every infinite $A \subseteq \omega$.

- If the answer is YES, there is a constructive proof.
- For a Borel ideal $\mathcal{I}$, there is a continuous such φ iff $\mathcal{ED}_{fin} \leq_{KB} \mathcal{I}$.

The ideal $\mathcal{R}$



Locally minimal ideals?

Problem

Is there a tall Borel ideal $\mathcal{J}$ such that for any tall Borel ideal $\mathcal{I}$ there is a $\mathcal{I}$-positive set X such that $\mathcal{J} \leq_K \mathcal{I} \restriction X$? Call such an $\mathcal{J}$ locally minimal.

$\mathcal{R} =$ the ideal generated by cliques and free sets in the Random graph

- $\mathcal{R} \not\leq_K \mathcal{I}$ iff $\omega \to (\mathcal{I}^+)^2_2$, and
- $\mathcal{R}$ is locally minimal iff $\mathcal{I}^+ \not\to (\mathcal{I}^+)^2_2$.

Problem

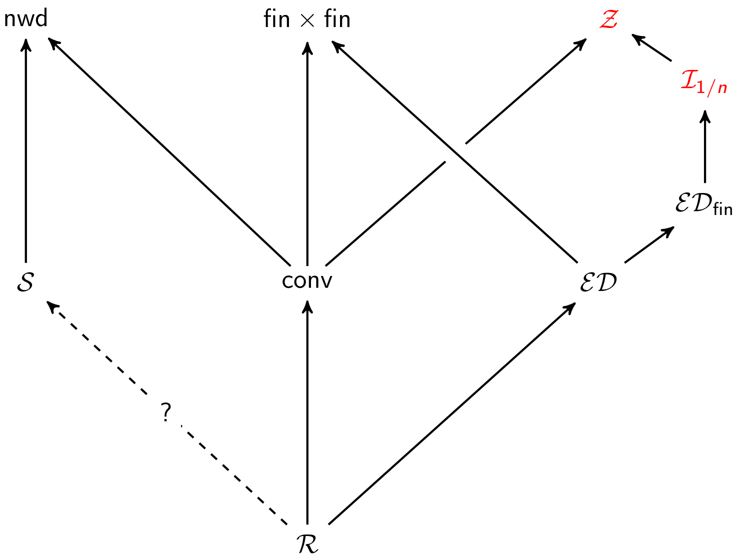
Is there a tall Borel (analytic) ideal $\mathcal{I}$ such that $\mathcal{I}^+ \to (\mathcal{I}^+)^2_2$?

- (H.-Meza-Thuemmel-Uzcategui '17) There are no such Borel ideals for which $\mathcal{P}(\omega)/\mathcal{I}$ is proper,
- (H.-Meza-Thuemmel-Uzcategui '17) There are no such F_σ-ideals
- (H.-Meza-Thuemmel-Uzcategui '17) There is a co-analytic ideal $\mathcal{I}$ such that $\mathcal{I}^+ \to (\mathcal{I}^+)^2_2$.

Analytic P-ideals

1. (Solecki '96) Every analytic P-ideal $\mathcal{I} = Exh(\varphi) = \{A \subseteq \omega : \lim_{n\to\infty} \varphi(A \setminus n) = 0\}$, for some lscs submeasure φ.
2. $\mathcal{ED}_{fin} \leq_K \mathcal{I}$ for every analytic P-ideal $\mathcal{I}$,
3. Every tall analytic P-ideal contains a tall summable ideal $\mathcal{I}_f = \{A \subseteq \omega : \sum_{n\in A} f(n) < \infty\}$.
4. (Sakai '16) There is a Katětov largest analytic P-ideal. Call it $\mathcal{P}_{max}$.
5. (Sakai '16) Moreover, all F_σ-ideals are below $\mathcal{P}_{max}$ in the Katětov order.

Analytic P-ideals



Measure dichotomy

$$\mathcal{Z} = \{A \subseteq \omega : \lim_{n\to\infty} \frac{|A \cap n|}{n} = 0\}$$

Let $\Omega = \{U \in Clop(2^\omega) : \mu(U) = \frac{1}{2}\}$.

$$\mathcal{S} = \{A \subseteq \Omega : \ \exists F \in [2^\omega]^{<\omega} \ \forall U \in A \ U \cap F \neq \emptyset\}$$

Theorem (H. '17)

Given an analytic P-ideal $\mathcal{I}$, either $\mathcal{I} \leq_K \mathcal{Z}$ or $\exists X \in \mathcal{I}^+ \ \mathcal{S} \leq_K \mathcal{I} \restriction X$.

1. (Solecki '00) $\exists X \in \mathcal{I}^+ \ \mathcal{S} \leq_K \mathcal{I} \restriction X$ iff $\mathcal{I}$ fails to satisfy Fatou's Lemma, equiv. has the Fubini property, i.e. $\forall \varepsilon > 0 \ \forall A \subseteq \omega \times 2^\omega$ Borel $(\{n < \omega : \mu(A_n) > \varepsilon\} \in \mathcal{I}^+ \Rightarrow \lambda^*(\{x \in 2^\omega : A^x \in \mathcal{I}^+\}) \geq \varepsilon)$.
2. $\mathcal{Z}$ is the Katětov largest Fubini analytic P-ideal.

Solecki ideal

$$\mathcal{S} = \{A \subseteq \Omega : \ \exists F \in [2^\omega]^{<\omega} \ \forall U \in A \ U \cap F \neq \emptyset\},$$

where $\Omega = \{U \in Clop(2^\omega) : \mu(U) = \frac{1}{2}\}$.

Problem

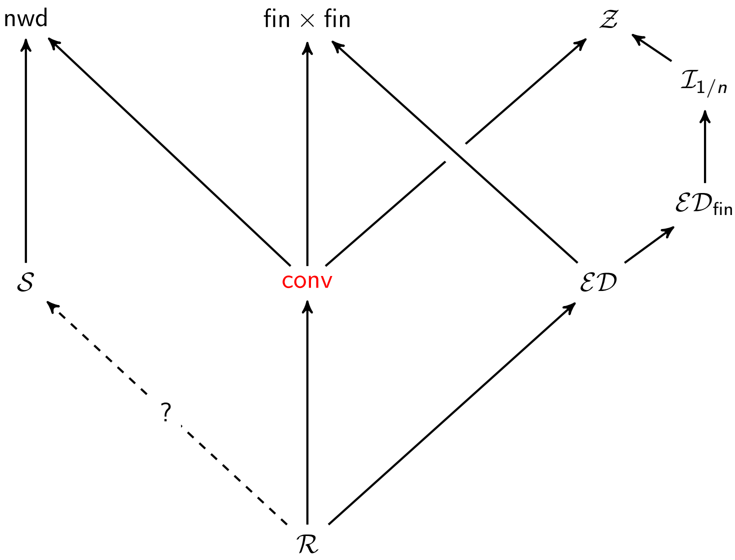
Is $\mathcal{R} \leq \mathcal{S}$? Equivalently, does $\omega \to (\mathcal{S}^+)^2_2$?

Problem

Given a $k > 1$, is there an $N > 1$ such that

$\forall c : [[2N]^N]^2 \to 2 \ \exists H \subseteq [2N]^N \ c-hom. \ \forall A \in [2N]^k \ \exists B \in H \ B \cap A = \emptyset$?

The ideal conv



The ideal *conv* and extensions of ideals

$$\text{conv} = \{A \subseteq \mathbb{Q} \cap [0,1] : \overline{A} \setminus Iso(A)| < \omega\}.$$

1. $conv \leq_K \mathcal{I}$ iff
$\exists \{A_n : n \in \omega\} \subseteq [\omega]^\omega \ \forall X \in \mathcal{I}^+ \ \exists n \in \omega \ |A_n \cap X| = |X \setminus A_n| = \omega.$

Problem

Can a Borel ideal $\mathcal{I}$ be extended to an F_σ-ideal if and only if $conv \not\leq_K \mathcal{I}$?

1. (H.-Meza-Uzcategui-Thuemmel '17) YES if, moreover, $\mathcal{P}(\omega)/\mathcal{I}$ is proper.
2. (Laflamme '92) Any ideal that can be extended to an F_σ-ideal can be destroyed by an ω^ω-bounding forcing.

Problem

Can every $F_{\sigma\delta}$-ideal be destroyed by a proper forcing not adding a dominating real? Can $\mathcal{Z}$? $\mathcal{P}_{max}$?

Upward cones and classification of ultrafilters

Definition (Baumgartner '95).

Let $\mathcal{U}$ be an ultrafilter and $\mathcal{I}$ an ideal on ω. We say that $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter if $\forall f : \omega \to \omega\ \exists U \in \mathcal{U}\ f[U] \in \mathcal{I}$.

... equivalent to $\mathcal{I} \not\leq_K \mathcal{U}^*$.

An ultrafilter $\mathcal{U}$ is

- selective iff $\mathcal{ED} \not\leq_K \mathcal{U}^*$ iff $\mathcal{R} \not\leq_K \mathcal{U}^*$
- a P-point iff $\mathit{fin} \times \mathit{fin} \not\leq_K \mathcal{U}^*$ iff $\mathit{conv} \not\leq_K \mathcal{U}^*$
- a Q-point iff $\mathcal{ED}_{\mathit{fin}} \not\leq_{KB} \mathcal{U}^*$,
- rapid iff $\mathcal{I} \not\leq_{KB} \mathcal{U}^*$ for any analytic P-ideal,
- Fubini (or Fatou or property (M)) iff $\mathcal{S} \not\leq_K \mathcal{U}^*$
- Hausdorff iff $\mathcal{G}_{FC} \not\leq_K \mathcal{U}^*$, ...

How trivial can (this) classification of ultrafilters be?

- (Isbell '65) Is it consistent that all ultrafilters have the same Tukey type?
- (Miller '80) Is it consistent that there are no P-points and no Q-points?
- Is it consistent that all ultrafilters* are in the Katětov order above all Borel ideals?

How trivial can this classification of ultrafilters be?

- (Pospíšil '39) There is an analytic ideal $\mathcal{I}$ and an ultrafilter $\mathcal{U}$ such that $\mathcal{I} \not\leq_K \mathcal{U}^*$.
- (Sakai '18) There is a Borel ideal $\mathcal{I}$ and an ultrafilter $\mathcal{U}$ such that $\mathcal{I} \not\leq_K \mathcal{U}^*$.
- (Guzmán-H. '20) There is an $F_{\sigma\delta\sigma}$-ideal $\mathcal{I}$ and an ultrafilter $\mathcal{U}$ such that $\mathcal{I} \not\leq_K \mathcal{U}^*$.
- In fact, the $\mathcal{I}$-ultrafilter $\mathcal{U}$ in the statement exists generically, i.e. every filter of character $< \mathfrak{c}$ can be extended to an $\mathcal{I}$-ultrafilter. This result is sharp:
- **(Guzmán-H. '20) It is consistent that for every $F_{\sigma\delta}$-ideal $\mathcal{I}$ there is a filter of character $< \mathfrak{c}$ which cannot be extended to an $\mathcal{I}$-ultrafilter.**

- **(Cancino '21) It is consistent that $\mathcal{I} \leq_{KB} \mathcal{U}^*$ for every F_σ-ideal $\mathcal{I}$ and every ultrafilter $\mathcal{U}$.**

How trivial can this classification of ultrafilters be?

- (Pospíšil '39) There is an analytic ideal $\mathcal{I}$ and an ultrafilter $\mathcal{U}$ such that $\mathcal{I} \not\leq_K \mathcal{U}^*$.
- (Sakai '18) There is a Borel ideal $\mathcal{I}$ and an ultrafilter $\mathcal{U}$ such that $\mathcal{I} \not\leq_K \mathcal{U}^*$.
- (Guzmán-H. '20) There is an $F_{\sigma\delta\sigma}$-ideal $\mathcal{I}$ and an ultrafilter $\mathcal{U}$ such that $\mathcal{I} \not\leq_K \mathcal{U}^*$.
- In fact, the $\mathcal{I}$-ultrafilter $\mathcal{U}$ in the statement exists generically, i.e. every filter of character $< \mathfrak{c}$ can be extended to an $\mathcal{I}$-ultrafilter. This result is sharp:
- **(Guzmán-H. '20) It is consistent that for every $F_{\sigma\delta}$-ideal $\mathcal{I}$ there is a filter of character $< \mathfrak{c}$ which cannot be extended to an $\mathcal{I}$-ultrafilter.**

- **(Cancino '21) It is consistent that $\mathcal{I} \leq_{KB} \mathcal{U}^*$ for every F_σ-ideal $\mathcal{I}$ and every ultrafilter $\mathcal{U}$.**

How trivial can this classification of ultrafilters be?

- (Pospíšil '39) There is an analytic ideal $\mathcal{I}$ and an ultrafilter $\mathcal{U}$ such that $\mathcal{I} \not\leq_K \mathcal{U}^*$.
- (Sakai '18) There is a Borel ideal $\mathcal{I}$ and an ultrafilter $\mathcal{U}$ such that $\mathcal{I} \not\leq_K \mathcal{U}^*$.
- (Guzmán-H. '20) There is an $F_{\sigma\delta\sigma}$-ideal $\mathcal{I}$ and an ultrafilter $\mathcal{U}$ such that $\mathcal{I} \not\leq_K \mathcal{U}^*$.
- In fact, the $\mathcal{I}$-ultrafilter $\mathcal{U}$ in the statement exists generically, i.e. every filter of character $< \mathfrak{c}$ can be extended to an $\mathcal{I}$-ultrafilter. This result is sharp:
- **(Guzmán-H. '20) It is consistent that for every $F_{\sigma\delta}$-ideal $\mathcal{I}$ there is a filter of character $< \mathfrak{c}$ which cannot be extended to an $\mathcal{I}$-ultrafilter.**

- **(Cancino '21) It is consistent that $\mathcal{I} \leq_{KB} \mathcal{U}^*$ for every F_σ-ideal $\mathcal{I}$ and every ultrafilter $\mathcal{U}$.**

How trivial can this classification of ultrafilters be?

Theorem (Cancino '21)

It is consistent that $\mathcal{I} \leq_{KB} \mathcal{U}^*$ for every F_σ-ideal $\mathcal{I}$ and ultrafilter $\mathcal{U}$.

In the model:

- There are no Hausdorff ultrafilters ((Choquet '69) $\mathcal{U}$ is Hausdorff if $\forall f, g \in \omega^\omega \; \exists U \in \mathcal{U} \; f \restriction U = g \restriction U$ or $f[U] \cap g[U] = \emptyset$.)
- There are no Fubini ultrafilters, but
- NCF holds in the model, so there are P-points.

What about $F_{\sigma\delta}$- ideals?

- Is there an $\mathcal{I}$-ultrafilter for some $F_{\sigma\delta}$- ideal $\mathcal{I}$?
- Is there a $\mathcal{Z}$-ultrafilter?
  - (Gryzlov '86) There is (in ZFC) an ultrafilter $\mathcal{U}$ such that for every one-to-one $f : \omega \to \omega$ there is a $U \in \mathcal{U}$ with $f[U] \in \mathcal{Z}$.
  - Assuming $\mathfrak{c} \leq \omega_2$ there is an ultrafilter $\mathcal{U}$ such that for every finite-to-one $f : \omega \to \omega$ there is a $U \in \mathcal{U}$ with $f[U] \in \mathcal{Z}$.
- (Miller '09) Is there a Sacks-indestructible ultrafilter?
  - (Chodounský-Guzmán-H. '21) If $\mathcal{U}$ is Sacks-indestructible ultrafilter then $\mathcal{U}$ is a $\mathcal{Z}$-ultrafilter.

What about $F_{\sigma\delta}$- ideals?

- Is there an $\mathcal{I}$-ultrafilter for some $F_{\sigma\delta}$- ideal $\mathcal{I}$?
- Is there a $\mathcal{Z}$-ultrafilter?
  - (Gryzlov '86) There is (in ZFC) an ultrafilter $\mathcal{U}$ such that for every one-to-one $f : \omega \to \omega$ there is a $U \in \mathcal{U}$ with $f[U] \in \mathcal{Z}$.
  - Assuming $\mathfrak{c} \leq \omega_2$ there is an ultrafilter $\mathcal{U}$ such that for every finite -to-one $f : \omega \to \omega$ there is a $U \in \mathcal{U}$ with $f[U] \in \mathcal{Z}$.
- (Miller '09) Is there a Sacks-indestructible ultrafilter?
  - (Chodounský-Guzmán-H. '21) If $\mathcal{U}$ is Sacks-indestructible ultrafilter then $\mathcal{U}$ is a $\mathcal{Z}$-ultrafilter.

Upside-down: MAD families and Katětov order

Many of the same questions one asks about ultrafilters can be asked about MAD families

- The ideals generated by MAD families are Katětov below $\mathit{fin} \times \mathit{fin}$.
- The ideals generated by MAD families are co-initial among tall ideals.
- (H.–Garcia-Ferreira '01) There are $\mathfrak{c}$-sized antichains and $\mathfrak{c}^+$-length decreasing chains of MAD families below any tall ideal.

Katětov maximal MAD families

A MAD family $\mathcal{A}$ is weakly tight if given a sequence $\{X_n : n \in \omega\} \subseteq \mathcal{I}^+(\mathcal{A})$ there is an $A \in \mathcal{A}$ such that $A \cap X_n$ is infinite for infinitely many $n \in \omega$.

- (H.–Garcia-Ferreira '01) A weakly tight MAD family $\mathcal{A}$ which is K-uniform (i.e K-equivalent to all restrictions) is **Katětov maximal**.
- (Raghavan-Steprāns '12) Assuming $\mathfrak{s} \leq \mathfrak{b}$ there is a weakly tight MAD family.
- (Arciga-H.-Martínez '13) Con(There is a Katětov maximal MAD family).
- (Guzmán '21) The existence of a Katětov-maximal MAD family follows from $non(\mathcal{M}) = \mathfrak{c}$, and $\diamondsuit(\mathfrak{d})$.
- Is there a weakly tight MAD family in ZFC?
- Is it consistent that there are no K-maximal (K-uniform) MAD families?
- There shouldn't be a Katětov-largest MAD family. We can only confirm this assuming (Guzmán '21: $\mathfrak{s} \leq \mathfrak{b}$ and a "version of" $\diamondsuit(\mathfrak{b})$).

Katětov maximal MAD families

A MAD family $\mathcal{A}$ is weakly tight if given a sequence $\{X_n : n \in \omega\} \subseteq \mathcal{I}^+(\mathcal{A})$ there is an $A \in \mathcal{A}$ such that $A \cap X_n$ is infinite for infinitely many $n \in \omega$.

- (H.–Garcia-Ferreira '01) A weakly tight MAD family $\mathcal{A}$ which is K-uniform (i.e K-equivalent to all restrictions) is **Katětov maximal**.
- (Raghavan-Steprāns '12) Assuming $\mathfrak{s} \leq \mathfrak{b}$ there is a weakly tight MAD family.
- (Arciga-H.-Martínez '13) Con(There is a Katětov maximal MAD family).
- (Guzmán '21) The existence of a Katětov-maximal MAD family follows from $non(\mathcal{M}) = \mathfrak{c}$, and $\diamondsuit(\mathfrak{d})$.
- Is there a weakly tight MAD family in ZFC?
- Is it consistent that there are no K-maximal (K-uniform) MAD families?
- There shouldn't be a Katětov-largest MAD family. We can only confirm this assuming (Guzmán '21: $\mathfrak{s} \leq \mathfrak{b}$ and a "version of" $\diamondsuit(\mathfrak{b})$).

Thank you for your attention!