Group representation for even and odd involutive commutative residuated chains

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The Main Result

- We shall present a one-to-one correspondence
  in a constructive manner
  between the class of all

  odd or even involutive FL_e-chains
  and the class of

  bunches of layer groups
  (certain direct systems of abelian *o*-groups)

Residuated lattices, FL-algebas

An algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, \mathsf{t}, \mathsf{f})$ is called a *full Lambek algebra* or an *FL-algebra*, if

- $(A, \wedge, \vee)$ is a lattice (i.e., $\wedge$, $\vee$ are commutative, associative and mutually absorptive),
- $(A, \cdot, \mathsf{t})$ is a monoid (i.e., $\cdot$ is associative, with unit element t),
- $x \cdot y \leq z$ iff $y \leq x \backslash z$ iff $x \leq z/y$, for all $x, y, z \in A$,
- f is an arbitrary element of A.

Residuated lattices are exactly the f-free reducts of FL-algebras. So, for an FL-algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, \mathsf{t}, \mathsf{f})$, the algebra $\mathbf{A}_r = (A, \wedge, \vee, \cdot, \backslash, /, \mathsf{t})$ is a residuated lattice and f is an arbitrary element of A. The maps $\backslash$ and $/$ are called the *left* and *right division*.

- commutative: $x \to y$

Odd or even involutive FL_e-chains

- FL_e-algebra : $\cdot$ is commutative
- FL_e-chain : $\leq$ is a linear order
- involutive : $x'' = x$ where $x' = x \to f$
- odd : $t = f$
- even : $x < t \Rightarrow x \leq f$

Odd or even involutive FL_e-chains

- MV-algebras - divisibility = IMTL-algebras
- $t' = f$
  integrality => t is in one of its extremal positions
  odd or even => t is in the other extremal position
- IMTL-algebras are semilinear (IMTL-chains)
- Our class is the non-integral analogue of IMTL-chains

Direct Systems

In mathematics, a **directed set** (or a **directed preorder** or a **filtered set**) is a nonempty set *A* together with a reflexive and transitive binary relation ≤ (that is, a preorder), with the additional property that every pair of elements has an upper bound.[1] In other words, for any *a* and *b* in *A* there must exist *c* in *A* with *a* ≤ *c* and *b* ≤ *c*. A directed set's preorder is called a *direction*.

Let $\langle I, \leq \rangle$ be a directed partially ordered set (note that not all authors require I to be directed). Let $A_\bullet = (A_i)_{i \in I}$ be a family of objects indexed by I and $f_{ij} \colon A_i \to A_j$ be a homomorphism for all $i \leq j$ with the following properties:

1. f_{ii} is the identity of A_i, and
2. **Compatibility condition**: $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$; that is,
$$A_i \xrightarrow{f_{ij}} A_j \xrightarrow{f_{jk}} A_k \quad \text{is equal to} \quad A_i \xrightarrow{f_{ik}} A_k.$$

Then the pair $\langle A_\bullet, f_{ij} \rangle$ is called a **direct system over** I. The maps f_{ij} are called the **bonding**, **connecting**, **transition**, or **linking maps/morphisms** of the system. If the bonding maps are understood or if there is no need to assign them symbols (e.g. as in the statements of some theorems) then the bonding maps will often be omitted (i.e. not written); for this reason it is common to see statements such as "let $A_\bullet$ be a direct system."

A bunch of layer groups

$$\left\langle \boldsymbol{G}_u, \boldsymbol{H}_u, \varsigma_{u \to v} \right\rangle_{\langle \Omega, \Psi, \Theta, \leq_{\kappa} \rangle}$$

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$$\left\langle \boldsymbol{G}_u\,,\, \boldsymbol{H}_u\,,\, \varsigma_{u \to v} \right\rangle \langle \Omega, \Psi, \Theta, \leq_{\kappa} \rangle$$

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$$\boldsymbol{G}_u = (G_u, \preceq_u, \cdot_u, {}^{-1_u}, u)$$

Bunches of layer groups

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Ω
{t}
∅
∅

Bunches of layer groups

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Ω	$\emptyset$	$\emptyset$
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for $u \in \Psi$, $\boldsymbol{G}_u$ is discrete, for $u \in \Psi$, $\varsigma_{u\to v}(u) = \varsigma_{u\to v}(u_{\downarrow u})$.

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for $u \in \Theta$, $\boldsymbol{H}_u \leq \boldsymbol{G}_u$

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for $u \in \Theta$, $\boldsymbol{H}_u \leq \boldsymbol{G}_u$

for $v \in \Theta$, $\varsigma_{u \to v}$ maps into H_v

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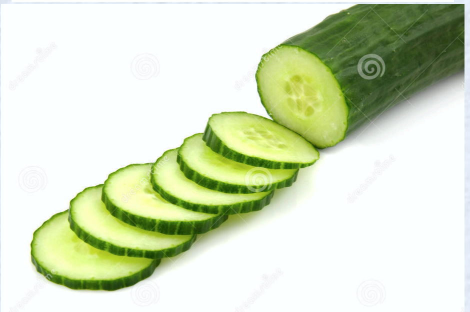
One-to-one correspondences

- the class of odd or even involutive FL_e-chains

$\mathbf{X}$

- the class of bunches of layer algebras (certain direct systems of more specific odd or even involutive FL_e-chains)

$\langle \mathbf{X}_u, \rho^{u\to v} \rangle_{\langle \Omega, \Psi, \Theta, \leq_\kappa \rangle}$



- the class of bunches of layer groups (certain direct systems of abelian *o*-groups)

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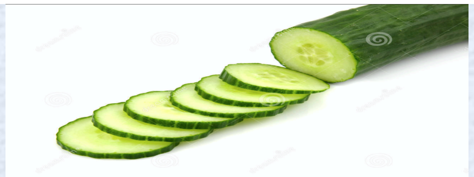

Lemma 5.2. *The following statements hold true.*

(1) *Given an odd or an even involutive FL_e-chain $\mathbf{X} = (X, \leq, \circledast, \to_\circledast, t, f)$ with residual complement operation $'$, $\mathcal{A}_{\mathbf{X}} = \langle \mathbf{X}_u, \rho^{u\to v} \rangle_{\langle \Omega, \Psi, \Theta, \leq_\kappa \rangle}$ is a bunch of layer algebras, called the* bunch of layer algebras *of* $\mathbf{X}$, *where $\tau(x) = x \to_\circledast x$, $\kappa = \{\tau(x) : x \in X\}$, $\leq_\kappa = \leq \cap (\kappa \times \kappa)$, $\kappa_I = \{u \in \kappa \setminus \{t\} : u' \text{ is idempotent}\}$, $\kappa_J = \{u \in \kappa \setminus \{t\} : u' \text{ is not idempotent}\}$, Ω, Ψ, Θ are defined by*

Ω	Ψ	Θ	
$\{t\}$	κ_J	κ_I	*if* $\mathbf{X}$ *is odd*
$\emptyset$	$\kappa_J \cup \{t\}$	κ_I	*if* $\mathbf{X}$ *is even and f is not idempotent*
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for $u \in \kappa$,

(5.3) $$\mathbf{X}_u = (X_u, \leq_u, \circledast_u, \to_{\circledast_u}, u, u'^{\mu}),$$

where $X_u = \{x \in X : \tau(x) = u\}$, $\leq_u = \leq \cap (X_u \times X_u)$, $\circledast_u = \circledast_{|X_u \times X_u}$, $\to_{\circledast_u} = \to_{\circledast|X_u \times X_u}$, for $x \in X_u$, $x'^{\mu} = x \to_\circledast u'$, and for $u, v \in \kappa$, $u <_\kappa v$, $\rho^{u\to v} : X_u \to X_v$ is given by

(5.4) $$\rho^{u\to v}(x) = v \circledast x.$$

(2) *Given a bunch of layer algebras $\mathcal{A} = \langle \mathbf{X}_u, \rho^{u\to v} \rangle_{\langle \Omega, \Psi, \Theta, \leq_\kappa \rangle}$, $\mathbf{X}_u = (X_u, \leq_u, \circledast_u, \to_{\circledast_u}, u, u'^{\mu})$ with $x'^{\mu} = x \to_{\circledast_u} u'^{\mu}$, $\mathcal{X}_{\mathcal{A}} = (X, \leq, \circledast, \to_\circledast, t, t')$ is an involutive FL_e-chain, called the* involutive FL_e-chain derived from $\mathcal{A}$, *where*

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by denoting for $u, v \in \kappa$, $uv = \max_\kappa(u, v)$ for short, for $x \in X_u$ and $y \in X_v$,

(5.7) $$x \leq y \text{ iff } \begin{cases} u = v \text{ and } x \leq_u y \\ u <_\kappa v \text{ and } \rho_v(x) \leq_v y \\ u >_\kappa v \text{ and } x <_u \rho_u(y) \end{cases},$$

(5.8) $$x \circledast y = \rho_{uv}(x) \circledast_{uv} \rho_{uv}(y),$$

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and t is the least element of κ. $\mathcal{X}_{\mathcal{A}}$ is odd if $t \in \Omega$, even with a non-idempotent falsum if $t \in \Psi$, and even with an idempotent falsum if $t \in \Theta$.

(3) *For a bunch of layer algebras $\mathcal{A}$, $\mathcal{A}_{(\mathcal{X}_{\mathcal{A}})} = \mathcal{A}$, and for an odd or even involutive FL_e-chain $\mathbf{X}$, $\mathcal{X}_{\mathcal{A}_{\mathbf{X}}} = \mathbf{X}$.*

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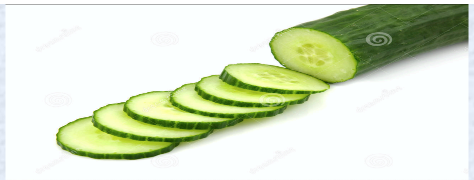

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One-to-one correspondences

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for $u \in \kappa$,

$$\mathbf{X}_u = (X_u, \leq_u, \circledast_u, \to_{\circledast_u}, u, u^{'^{u}}), \tag{5.3}$$

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$$X = \dot{\bigcup}_{u\in\kappa} X_u, \tag{5.5}$$

for $v \in \kappa$, $\rho_v : X \to X$ is defined by

$$\rho_v(x) = \begin{cases} \rho^{u\to v}(x) & \text{if } u <_\kappa v \text{ and } x \in X_u \\ x & \text{if } u \geq_\kappa v \text{ and } x \in X_u \end{cases}, \tag{5.6}$$

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$$x \to_\circledast y = (x \circledast y')', \tag{5.10}$$

and t is the least element of κ. $\mathcal{X}_{\mathcal{A}}$ is odd if $t \in \Omega$, even with a non-idempotent falsum if $t \in \Psi$, and even with an idempotent falsum if $t \in \Theta$.

(3) For a bunch of layer algebras $\mathcal{A}$, $\mathcal{A}_{(\mathcal{X}_{\mathcal{A}})} = \mathcal{A}$, and for an odd or even involutive FL_e-chain $\mathbf{X}$, $\mathcal{X}_{\mathcal{A}_{\mathbf{X}}} = \mathbf{X}$.

(2) Given a bunch of layer groups $\mathcal{G} = \langle \boldsymbol{G}_u, \boldsymbol{H}_u, \varsigma_{u\to v}\rangle_{\boldsymbol{\kappa}}$ with $\boldsymbol{\kappa} = \langle \Omega, \Psi, \Theta, \leq_\kappa\rangle$, $\mathcal{A}_{\mathcal{G}} = \langle \mathbf{X}_u, \rho_{u\to v}\rangle_{\boldsymbol{\kappa}}$ is bunch of layer algebras, called the bunch of layer algebras derived from *$\mathcal{G}$, where*

$$\mathbf{X}_u = (X_u, \leq_u, \cdot_u, \to_u, u, u^{'^{u}}) = \begin{cases} \iota(\boldsymbol{G}_u) & \text{if } u \in \Omega \\ \iota(\boldsymbol{G}_u)_{\downarrow} & \text{if } u \in \Psi \\ Sp(\iota(\boldsymbol{G}_u), \iota(\boldsymbol{H}_u)), & \text{if } u \in \Theta \end{cases}, \tag{7.5}$$

$\kappa = \Omega \cup \Psi \cup \Theta$, and for $u, v \in \kappa$ such that $u \leq_\kappa v$, $\rho_{u\to v} : X_u \to X_v$ is defined by

$$\rho_{u\to v} = \begin{cases} \varsigma_{u\to v} & \text{if } u \notin \Theta \\ \varsigma_{u\to v} \circ h_u & \text{if } v > u \in \Theta \\ id_{X_u} & \text{if } v = u \in \Theta \end{cases}, \tag{7.6}$$

where h_u is the canonical homomorphism of $\mathbf{X}_u$.

One-to-one correspondences

- the class of odd or even involutive FL_e-chains

X

Lemma 5.2. *The following statements hold true.*

(1) Given an odd or an even involutive FL_e-chain $\mathbf{X} = (X, \leq, \circledast, \rightarrow_\circledast, t, f)$ with residual complement operation $'$, $\mathcal{A}_\mathbf{X} = \langle \mathbf{X}_u, \rho^{u\to v} \rangle_{\langle \Omega, \Psi, \Theta, \leq_\kappa \rangle}$ is a bunch of layer algebras, called the bunch of layer algebras *of* $\mathbf{X}$, *where $\tau(x) = x \rightarrow_\circledast x$, $\kappa = \{\tau(x) : x \in X\}$, $\leq_\kappa = \leq \cap (\kappa \times \kappa)$, $\kappa_I = \{u \in \kappa \setminus \{t\} : u' \text{ is idempotent}\}$, $\kappa_J = \{u \in \kappa \setminus \{t\} : u' \text{ is not idempotent}\}$, Ω, Ψ, Θ are defined by*

Ω	Ψ	Θ	
$\{t\}$	κ_J	κ_I	*if* $\mathbf{X}$ *is odd*
$\emptyset$	$\kappa_J \cup \{t\}$	κ_I	*if* $\mathbf{X}$ *is even and f is not idempotent*

(2) Given a bunch of layer algebras $\mathcal{A} = \langle \mathbf{X}_u, \rho^{u\to v} \rangle_{\langle \Omega, \Psi, \Theta, \leq_\kappa \rangle}$, $\mathbf{X}_u = (X_u, \leq_u, \circledast_u, \rightarrow_{\circledast_u}, u, u'^{\mu})$ with $x'^{\mu} = x \rightarrow_{\circledast_u} u'^{\mu}$, $\mathcal{X}_\mathcal{A} = (X, \leq, \circledast, \rightarrow_\circledast, t, t')$ is an involutive FL_e-chain, called the involutive FL_e-chain derived from $\mathcal{A}$, *where*

$$X = \dot{\bigcup}_{u \in \kappa} X_u, \tag{5.5}$$

for $v \in \kappa$, $\rho_v : X \to X$ is defined by

$$\rho_v(x) = \begin{cases} \rho^{u\to v}(x) & \text{if } u <_\kappa v \text{ and } x \in X_u \\ x & \text{if } u \geq_\kappa v \text{ and } x \in X_u \end{cases}, \tag{5.6}$$

by denoting for $u, v \in \kappa$, $uv = \max_\kappa(u, v)$ for short, for $x \in X_u$ and $y \in X_v$,

(2) Given a bunch of layer groups $\mathcal{G} = \langle \boldsymbol{G}_u, \boldsymbol{H}_u, \varsigma_{u\to v} \rangle_{\boldsymbol{\kappa}}$ with $\boldsymbol{\kappa} = \langle \Omega, \Psi, \Theta, \leq_\kappa \rangle$, $\mathcal{A}_\mathcal{G} = \langle \mathbf{X}_u, \rho_{u\to v} \rangle_{\boldsymbol{\kappa}}$ is bunch of layer algebras, called the bunch of layer algebras derived from $\mathcal{G}$, *where*

$$\mathbf{X}_u = (X_u, \leq_u, \cdot_u, \rightarrow_u, u, u'^{u}) = \begin{cases} \iota(\boldsymbol{G}_u) & \text{if } u \in \Omega \\ \iota(\boldsymbol{G}_u)_{\downarrow} & \text{if } u \in \Psi \\ Sp(\iota(\boldsymbol{G}_u), \iota(\boldsymbol{H}_u)), & \text{if } u \in \Theta \end{cases}, \tag{7.5}$$

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where h_u is the canonical homomorphism of $\mathbf{X}_u$.

(2) *Given a bunch of layer algebras* $\mathcal{A} = \langle \mathbf{X}_u, \rho^{u \to v} \rangle_{\langle \Omega, \Psi, \Theta, \leq_\kappa \rangle}$, $\mathbf{X}_u = (X_u, \leq_u, \circledast_u, \to_{\circledast_u}, u, u^{'^{u}})$ *with* $x^{'^{u}} = x \to_{\circledast_u} u^{'^{u}}$, $\mathcal{X}_\mathcal{A} = (X, \leq, \circledast, \to_\circledast, t, t')$ *is an involutive* FL_e*-chain, called the* involutive FL_e-chain derived from $\mathcal{A}$, *where*

$$X = \dot{\bigcup}_{u \in \kappa} X_u, \tag{5.5}$$

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by denoting for $u, v \in \kappa$, $uv = \max_\kappa(u, v)$ *for short, for* $x \in X_u$ *and* $y \in X_v$,

$$x \leq y \text{ iff } \begin{cases} u = v \text{ and } x \leq_u y \\ u <_\kappa v \text{ and } \rho_v(x) \leq_v y \\ u >_\kappa v \text{ and } x <_u \rho_u(y) \end{cases}, \tag{5.7}$$

$$x \circledast y = \rho_{uv}(x) \circledast_{uv} \rho_{uv}(y), \tag{5.8}$$

$$x' = x^{'^{u}}, \tag{5.9}$$

$$x \to_\circledast y = (x \circledast y')', \tag{5.10}$$

and t *is the least element of* κ. $\mathcal{X}_\mathcal{A}$ *is odd if* $t \in \Omega$, *even with a non-idempotent falsum if* $t \in \Psi$, *and even with an idempotent falsum if* $t \in \Theta$.

(3) *For a bunch of layer algebras* $\mathcal{A}$, $\mathcal{A}_{(\mathcal{X}_\mathcal{A})} = \mathcal{A}$, *and for an odd or even involutive* FL_e*-chain* $\mathbf{X}$, $\mathcal{X}_{\mathcal{A}_\mathbf{X}} = \mathbf{X}$.

(certain direct systems
of abelian *o*-groups)

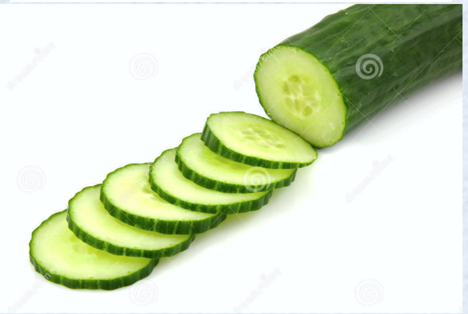
One-to-one correspondences

- the class of odd or even involutive $\mathrm{FL_e}$-chains

$\mathbf{X}$

- the class of bunches of layer algebras

$\langle \mathbf{X}_u, \rho^{u\to v}\rangle_{\langle\Omega,\Psi,\Theta,\leq_\kappa\rangle}$



Definition 5.1. Let $(\kappa,\leq_\kappa)$ be a totally ordered set with least element t, and let an ordered triple $\langle\kappa_I,\kappa_J,\{t\}\rangle$ be a partition of κ, where κ_I and κ_J can also be empty. Define Ω, Ψ, and Θ by one of the rows of

Ω	Ψ	Θ
$\{t\}$	κ_J	κ_I
$\emptyset$	$\kappa_J\cup\{t\}$	κ_I
$\emptyset$	κ_J	$\kappa_I\cup\{t\}$

Let $\mathbf{X}_u=(X_u,\leq_u,\circledast_u,\to_{\circledast_u},u,u^{\prime_\mu})$ be a family of involutive FL_e-chains indexed by elements of κ (denote ${}^{\prime_\mu}$ the respective residual complement operations), such that $\mathbf{X}_u$ is

$$(5.1)\qquad \begin{cases} \text{cancellative and odd} & \text{if } u\in\Omega\\ \text{discretely ordered, cancellative and even}^{8} & \text{if } u\in\Psi\\ \text{even with an idempotent falsum satisfying } x\circledast_u x^{\prime_\mu}=u^{\prime_\mu} & \text{if } u\in\Theta \end{cases},$$

and such that for $u,v\in\kappa$, $u\leq_\kappa v$, there exist a

$$(5.2)\qquad \text{homomorphism } \rho^{u\to v}$$

from the residuated lattice reduct of $\mathbf{X}_u$ to the residuated lattice reduct of $\mathbf{X}_v$ satisfying

(A1) $\rho^{v\to w}\circ\rho^{u\to v}=\rho^{u\to w}$ (direct system property),
(A2) for $u\notin\Omega$, $\rho^{u\to v}(u)=\rho^{u\to v}(u^{\prime_\mu})$.

Call $\mathcal{A}=\langle\mathbf{X}_u,\rho^{u\to v}\rangle_{\langle\Omega,\Psi,\Theta,\leq_\kappa\rangle}$ a *bunch of layer algebras*. Call the $\mathbf{X}_u$'s the layer algebras, call $\langle\kappa,\leq_\kappa\rangle$ the *skeleton*, call $\langle\Omega,\Psi,\Theta\rangle$ the *partition* of the skeleton, and call $\langle\mathbf{X}_u,\rho_{u\to v}\rangle_\kappa$ the direct system of layer algebras over κ. Note that κ can be recovered from its partition, (and ultimately, from $\mathcal{A}$) via $\kappa=\Omega\cup\Psi\cup\Theta$.

[8] Hence with a non-idempotent falsum.

Lemma 5.2. *The following statements hold true.*

(1) *Given an odd or an even involutive FL_e-chain $\mathbf{X}=(X,\leq,\circledast,\to_\circledast,t,f)$ with residual complement operation $'$, $\mathcal{A}_\mathbf{X}=\langle\mathbf{X}_u,\rho^{u\to v}\rangle_{\langle\Omega,\Psi,\Theta,\leq_\kappa\rangle}$ is a bunch of layer algebras, called the* bunch of layer algebras *of $\mathbf{X}$, where $\tau(x)=x\to_\circledast x$, $\kappa=\{\tau(x):x\in X\}$, $\leq_\kappa=\leq\cap(\kappa\times\kappa)$, $\kappa_I=\{u\in\kappa\setminus\{t\}:u' \text{ is idempotent}\}$, $\kappa_J=\{u\in\kappa\setminus\{t\}:u' \text{ is not idempotent}\}$, Ω, Ψ, Θ are defined by*

Ω	Ψ	Θ	
$\{t\}$	κ_J	κ_I	*if $\mathbf{X}$ is odd*
$\emptyset$	$\kappa_J\cup\{t\}$	κ_I	*if $\mathbf{X}$ is even and f is not idempotent*
$\emptyset$	κ_J	$\kappa_I\cup\{t\}$	*if $\mathbf{X}$ is even and f is idempotent*

for $u\in\kappa$,

$$(5.3)\qquad \mathbf{X}_u=(X_u,\leq_u,\circledast_u,\to_{\circledast_u},u,u^{\prime_\mu}),$$

where $X_u=\{x\in X:\tau(x)=u\}$, $\leq_u=\leq\cap(X_u\times X_u)$, $\circledast_u=\circledast_{|X_u\times X_u}$, $\to_{\circledast_u}=\to_{\circledast|X_u\times X_u}$, for $x\in X_u$, $x^{\prime_\mu}=x\to_\circledast u'$, and for $u,v\in\kappa$, $u<_\kappa v$, $\rho^{u\to v}:X_u\to X_v$ is given by

$$(5.4)\qquad \rho^{u\to v}(x)=v\circledast x.$$

(2) *Given a bunch of layer algebras $\mathcal{A}=\langle\mathbf{X}_u,\rho^{u\to v}\rangle_{\langle\Omega,\Psi,\Theta,\leq_\kappa\rangle}$, $\mathbf{X}_u=(X_u,\leq_u,\circledast_u,\to_{\circledast_u},u,u^{\prime_\mu})$ with $x^{\prime_\mu}=x\to_{\circledast_u}u^{\prime_\mu}$, $\mathcal{X}_\mathcal{A}=(X,\leq,\circledast,\to_\circledast,t,t')$ is an involutive FL_e-chain, called the* involutive FL_e-chain derived from *$\mathcal{A}$, where*

$$(5.5)\qquad X=\dot{\bigcup}_{u\in\kappa}X_u,$$

for $v\in\kappa$, $\rho_v:X\to X$ is defined by

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by denoting for $u,v\in\kappa$, $uv=\max_\kappa(u,v)$ for short, for $x\in X_u$ and $y\in X_v$,

$$(5.7)\qquad x\leq y \text{ iff } \begin{cases} u=v \text{ and } x\leq_u y\\ u<_\kappa v \text{ and } \rho_v(x)\leq_v y\\ u>_\kappa v \text{ and } x<_u\rho_u(y)\end{cases},$$

$$(5.8)\qquad x\circledast y=\rho_{uv}(x)\circledast_{uv}\rho_{uv}(y),$$

$$(5.9)\qquad x'=x^{\prime_\mu},$$

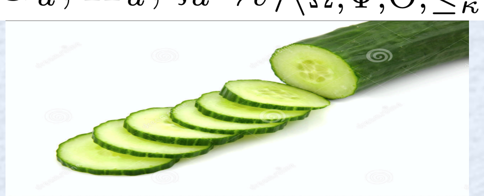
$$(5.10)\qquad x\to_\circledast y=(x\circledast y')',$$

and t is the least element of κ. $\mathcal{X}_\mathcal{A}$ is odd if $t\in\Omega$, even with a non-idempotent falsum if $t\in\Psi$, and even with an idempotent falsum if $t\in\Theta$.

(3) *For a bunch of layer algebras $\mathcal{A}$, $\mathcal{A}_{(\mathcal{X}_\mathcal{A})}=\mathcal{A}$, and for an odd or even involutive FL_e-chain $\mathbf{X}$, $\mathcal{X}_{\mathcal{A}_\mathbf{X}}=\mathbf{X}$.*

- the class of bunches of layer groups

$\langle \boldsymbol{G}_u, \boldsymbol{H}_u, \varsigma_{u\to v}\rangle_{\langle\Omega,\Psi,\Theta,\leq_\kappa\rangle}$



Definition 2.3. [4, Definition 6.1] Call $\mathcal{X}=\langle\boldsymbol{G}_u,\boldsymbol{H}_u,\varsigma_{u\to v}\rangle_{\langle\Omega,\Psi,\Theta,\leq_\kappa\rangle}$ a *bunch of layer groups*, where $(\kappa,\leq_\kappa)$ is a totally ordered set with least element t, the ordered triple $\langle\{t\},\kappa_J,\kappa_I\rangle$ is a partition of κ, where κ_I and κ_J can also be empty, Ω, Ψ, and Θ are defined by one of the rows of

Ω	Ψ	Θ
$\{t\}$	κ_J	κ_I
$\emptyset$	$\kappa_J\cup\{t\}$	κ_I
$\emptyset$	κ_J	$\kappa_I\cup\{t\}$

$\boldsymbol{G}_u=(G_u,\preceq_u,\cdot_u,{}^{-1_u},u)$ is a family of abelian o-groups indexed by elements of κ, and
$\boldsymbol{H}_u=(H_u,\preceq_u,\cdot_u,{}^{-1_u},u)$ is a family of abelian o-groups indexed by elements of Θ, such that

$$\text{for } u\in\Psi,\ \boldsymbol{G}_u \text{ is discrete,}$$

$$\text{for } u\in\Theta,\ \boldsymbol{H}_u\leq\boldsymbol{G}_u,$$

and such that for every $u,v\in\kappa$, $u\leq_\kappa v$, there exists a homomorphism $\varsigma_{u\to v}:G_u\to G_v$ satisfying

(G1) $\varsigma_{v\to w}\circ\varsigma_{u\to v}=\varsigma_{u\to w}$ (direct system property),
(G2) for $v\in\Theta$, $\varsigma_{u\to v}$ maps into H_v.
(G3) for $u\in\Psi$, $\varsigma_{u\to v}(u)=\varsigma_{u\to v}(u_{\downarrow u})$,

Call the $\boldsymbol{G}_u$'s and the $\boldsymbol{H}_u$'s the layer groups and layer subgroups of $\mathcal{X}$, respectively, call $\langle\kappa,\leq_\kappa\rangle$ the *skeleton* of $\mathcal{X}$, call $\langle\Omega,\Psi,\Theta\rangle$ the *partition* of the skeleton, and call $\langle\boldsymbol{G}_u,\varsigma_{u\to v}\rangle_\kappa$ the direct system of $\mathcal{X}$. Note that κ can be recovered from its partition, (and ultimately, from $\mathcal{X}$) via $\kappa=\Omega\cup\Psi\cup\Theta$.

Lemma 7.2. *The following statements hold true.*

(1) *Given a bunch of layer algebras $\mathcal{A}=\langle\mathbf{X}_u,\rho_{u\to v}\rangle_{\boldsymbol{\kappa}}$ with $\boldsymbol{\kappa}=\langle\Omega,\Psi,\Theta,\leq_\kappa\rangle$, $\mathcal{G}_\mathcal{A}=\langle\boldsymbol{G}_u,\boldsymbol{H}_u,\varsigma_{u\to v}\rangle_{\boldsymbol{\kappa}}$ is bunch of layer groups, where*

$$(7.2)\qquad \boldsymbol{G}_u=(G_u,\preceq_u,\cdot_u,{}^{-1_u},u)=\begin{cases}\iota(\mathbf{X}_u) & \text{if } u\in\Omega\\ \iota\left(\mathbf{X}_{u\uparrow}\right) & \text{if } u\in\Psi\\ \iota(\pi_1(\mathbf{X}_u)) & \text{if } u\in\Theta\end{cases},$$

for $u\in\Theta$,

$$(7.3)\qquad \boldsymbol{H}_u=(H_u,\preceq_u,\cdot_u,{}^{-1_u},u)=\iota(\pi_2(\mathbf{X}_u)),$$

$\kappa=\Omega\cup\Psi\cup\Theta$, and for $u,v\in\kappa$ such that $u\leq_\kappa v$, $\varsigma_{u\to v}:G_u\to G_v$ is defined by

$$(7.4)\qquad \varsigma_{u\to v}=\rho_{u\to v}|_{G_u}.$$

Call $\mathcal{G}_\mathcal{A}$ the bunch of layer groups derived from *$\mathcal{A}$.*

(2) *Given a bunch of layer groups $\mathcal{G}=\langle\boldsymbol{G}_u,\boldsymbol{H}_u,\varsigma_{u\to v}\rangle_{\boldsymbol{\kappa}}$ with $\boldsymbol{\kappa}=\langle\Omega,\Psi,\Theta,\leq_\kappa\rangle$, $\mathcal{A}_\mathcal{G}=\langle\mathbf{X}_u,\rho_{u\to v}\rangle_{\boldsymbol{\kappa}}$ is bunch of layer algebras, called the* bunch of layer algebras derived from *$\mathcal{G}$, where*

$$(7.5)\qquad \mathbf{X}_u=(X_u,\leq_u,\cdot_u,\to_u,u,u^{\prime_\mu})=\begin{cases}\iota(\boldsymbol{G}_u) & \text{if } u\in\Omega\\ \iota(\boldsymbol{G}_u)_\downarrow & \text{if } u\in\Psi\\ Sp(\iota(\boldsymbol{G}_u),\iota(\boldsymbol{H}_u)), & \text{if } u\in\Theta\end{cases},$$

$\kappa=\Omega\cup\Psi\cup\Theta$, and for $u,v\in\kappa$ such that $u\leq_\kappa v$, $\rho_{u\to v}:X_u\to X_v$ is defined by

$$(7.6)\qquad \rho_{u\to v}=\begin{cases}\varsigma_{u\to v} & \text{if } u\notin\Theta\\ \varsigma_{u\to v}\circ h_u & \text{if } v>u\in\Theta\\ id_{X_u} & \text{if } v=u\in\Theta\end{cases},$$

where h_u is the canonical homomorphism of $\mathbf{X}_u$.

Definition 5.1. Let $(\kappa, \leq_\kappa)$ be a totally ordered set with least element t, and let an ordered triple $\langle \kappa_I, \kappa_J, \{t\} \rangle$ be a partition of κ, where κ_I and κ_J can also be empty. Define Ω, Ψ, and Θ by one of the rows of

Ω	Ψ	Θ
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Let $\mathbf{X}_u = (X_u, \leq_u, \circledast_u, \rightarrow_{\circledast_u}, u, u'^{u})$ be a family of involutive FL$_e$-chains indexed by elements of κ (denote $'^{u}$ the respective residual complement operations), such that $\mathbf{X}_u$ is

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from the residuated lattice reduct of $\mathbf{X}_u$ to the residuated lattice reduct of $\mathbf{X}_v$ satisfying

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Call $\mathcal{A} = \langle \mathbf{X}_u, \rho^{u \to v} \rangle_{\langle \Omega, \Psi, \Theta, \leq_\kappa \rangle}$ a *bunch of layer algebras*. Call the $\mathbf{X}_u$'s the layer algebras, call $\langle \kappa, \leq_\kappa \rangle$ the *skeleton*, call $\langle \Omega, \Psi, \Theta \rangle$ the *partition* of the skeleton, and call $\langle \mathbf{X}_u, \rho_{u \to v} \rangle_\kappa$ the direct system of layer algebras over κ. Note that κ can be recovered from its partition, (and ultimately, from $\mathcal{A}$) via $\kappa = \Omega \cup \Psi \cup \Theta$.

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$\boldsymbol{G}_u = (G_u, \preceq_u, \cdot_u, {}^{-1_u}, u)$ is a family of abelian o-groups indexed by elements of κ, and
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$$\text{for } u \in \Psi,\ \boldsymbol{G}_u \text{ is discrete,}$$
$$\text{for } u \in \Theta,\ \boldsymbol{H}_u \leq \boldsymbol{G}_u,$$

and such that for every $u, v \in \kappa$, $u \leq_\kappa v$, there exists a homomorphism $\varsigma_{u \to v} : G_u \to G_v$ satisfying

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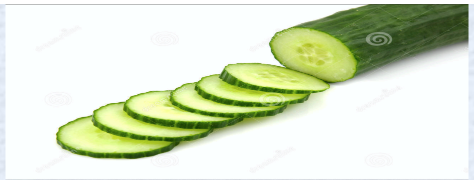

One-to-one correspondences

$$\langle \boldsymbol{G}_u, \boldsymbol{H}_u, \varsigma_{u\to v} \rangle_{\langle \Omega, \Psi, \Theta, \leq_\kappa \rangle}$$

Definition 2.3. [4, Definition 6.1] Call $\mathcal{X} = \langle \boldsymbol{G}_u, \boldsymbol{H}_u, \varsigma_{u\to v} \rangle_{\langle \Omega, \Psi, \Theta, \leq_\kappa \rangle}$ a *bunch of layer groups*, where $(\kappa, \leq_\kappa)$ is a totally ordered set with least element t, the ordered triple $\langle \{t\}, \kappa_J, \kappa_I \rangle$ is a partition of κ, where κ_I and κ_J can also be empty, Ω, Ψ, and Θ are defined by one of the rows of

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and such that for every $u, v \in \kappa$, $u \leq_\kappa v$, there exists a homomorphism $\varsigma_{u\to v} : G_u \to G_v$ satisfying

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(G2) for $v \in \Theta$, $\varsigma_{u\to v}$ maps into H_v.
(G3) for $u \in \Psi$, $\varsigma_{u\to v}(u) = \varsigma_{u\to v}(u_{\downarrow u})$,

Call the $\boldsymbol{G}_u$'s and the $\boldsymbol{H}_u$'s the layer groups and layer subgroups of $\mathcal{X}$, respectively, call $\langle \kappa, \leq_\kappa \rangle$ the *skeleton* of $\mathcal{X}$, call $\langle \Omega, \Psi, \Theta \rangle$ the *partition* of the skeleton, and call $\langle \boldsymbol{G}_u, \varsigma_{u\to v} \rangle_\kappa$ the direct system of $\mathcal{X}$. Note that κ can be recovered from its partition, (and ultimately, from $\mathcal{X}$) via $\kappa = \Omega \cup \Psi \cup \Theta$.

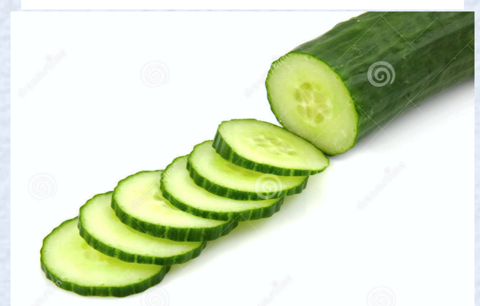
One-to-one correspondences

- the class of odd or even involutive $\mathrm{FL_e}$-chains

$\mathbf{X}$

the class of bunches of layer algebras (certain direct systems of more specific odd or even involutive $\mathrm{FL_e}$-chains)

$\langle \mathbf{X}_u, \rho^{u\to v} \rangle_{\langle \Omega,\Psi,\Theta,\leq_\kappa \rangle}$



Lemma 5.2. *The following statements hold true.*

(1) Given an odd or an even involutive FL_e-chain $\mathbf{X} = (X, \leq, \circledast, \to_\circledast, t, f)$ with residual complement operation $'$, $\mathcal{A}_\mathbf{X} = \langle \mathbf{X}_u, \rho^{u\to v}\rangle_{\langle\Omega,\Psi,\Theta,\leq_\kappa\rangle}$ is a bunch of layer algebras, called the bunch of layer algebras *of $\mathbf{X}$, where $\tau(x) = x \to_\circledast x$, $\kappa = \{\tau(x) : x \in X\}$, $\leq_\kappa = \leq \cap (\kappa\times\kappa)$, $\kappa_I = \{u \in \kappa \setminus \{t\} : u' \text{ is idempotent}\}$, $\kappa_J = \{u \in \kappa \setminus \{t\} : u' \text{ is not idempotent}\}$, Ω, Ψ, Θ are defined by*

Ω	Ψ	Θ	
$\{t\}$	κ_J	κ_I	*if $\mathbf{X}$ is odd*
$\emptyset$	$\kappa_J \cup \{t\}$	κ_I	*if $\mathbf{X}$ is even and f is not idempotent*
$\emptyset$	κ_J	$\kappa_I \cup \{t\}$	*if $\mathbf{X}$ is even and f is idempotent*

for $u \in \kappa$,

$$\mathbf{X}_u = (X_u, \leq_u, \circledast_u, \to_{\circledast_u}, u, u'^{\mu}), \tag{5.3}$$

where $X_u = \{x \in X : \tau(x) = u\}$, $\leq_u = \leq \cap (X_u \times X_u)$, $\circledast_u = \circledast_{|X_u\times X_u}$, $\to_{\circledast_u} = \to_{\circledast|X_u\times X_u}$, for $x \in X_u$, $x'^{\mu} = x \to_\circledast u'$, and for $u, v \in \kappa$, $u <_\kappa v$, $\rho^{u\to v} : X_u \to X_v$ is given by

$$\rho^{u\to v}(x) = v \circledast x. \tag{5.4}$$

(2) Given a bunch of layer algebras $\mathcal{A} = \langle \mathbf{X}_u, \rho^{u\to v}\rangle_{\langle\Omega,\Psi,\Theta,\leq_\kappa\rangle}$, $\mathbf{X}_u = (X_u, \leq_u, \circledast_u, \to_{\circledast_u}, u, u'^{\mu})$ with $x'^{\mu} = x \to_{\circledast_u} u'^{\mu}$, $\mathcal{X}_\mathcal{A} = (X, \leq, \circledast, \to_\circledast, t, t')$ is an involutive FL_e-chain, called the involutive FL_e-chain derived from $\mathcal{A}$, *where*

$$X = \dot{\bigcup}_{u\in\kappa} X_u, \tag{5.5}$$

for $v \in \kappa$, $\rho_v : X \to X$ is defined by

$$\rho_v(x) = \begin{cases} \rho^{u\to v}(x) & \text{if } u <_\kappa v \text{ and } x \in X_u \\ x & \text{if } u \geq_\kappa v \text{ and } x \in X_u \end{cases}, \tag{5.6}$$

by denoting for $u, v \in \kappa$, $uv = \max_\kappa(u, v)$ for short, for $x \in X_u$ and $y \in X_v$,

$$x \leq y \text{ iff } \begin{cases} u = v \text{ and } x \leq_u y \\ u <_\kappa v \text{ and } \rho_v(x) \leq_v y \\ u >_\kappa v \text{ and } x <_u \rho_u(y) \end{cases}, \tag{5.7}$$

$$x \circledast y = \rho_{uv}(x) \circledast_{uv} \rho_{uv}(y), \tag{5.8}$$

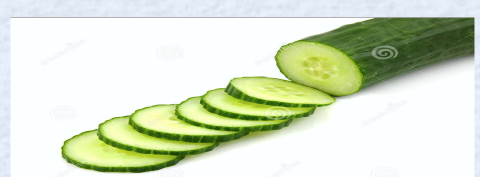
$$x' = x'^{\mu}, \tag{5.9}$$

$$x \to_\circledast y = (x \circledast y')', \tag{5.10}$$

and t is the least element of κ. $\mathcal{X}_\mathcal{A}$ is odd if $t \in \Omega$, even with a non-idempotent falsum if $t \in \Psi$, and even with an idempotent falsum if $t \in \Theta$.

(3) For a bunch of layer algebras $\mathcal{A}$, $\mathcal{A}_{(\mathcal{X}_\mathcal{A})} = \mathcal{A}$, and for an odd or even involutive FL_e-chain $\mathbf{X}$, $\mathcal{X}_{\mathcal{A}_\mathbf{X}} = \mathbf{X}$.

- the class of bunches of layer groups (certain direct systems of abelian *o*-groups)



Lemma 7.2. *The following statements hold true.*

(1) Given a bunch of layer algebras $\mathcal{A} = \langle \mathbf{X}_u, \rho_{u\to v}\rangle_{\boldsymbol{\kappa}}$ with $\boldsymbol{\kappa} = \langle \Omega, \Psi, \Theta, \leq_\kappa\rangle$, $\mathcal{G}_\mathcal{A} = \langle \boldsymbol{G}_u, \boldsymbol{H}_u, \varsigma_{u\to v}\rangle_{\boldsymbol{\kappa}}$ is bunch of layer groups, where

$$\boldsymbol{G}_u = (G_u, \preceq_u, \cdot_u, {}^{-1_u}, u) = \begin{cases} \iota(\mathbf{X}_u) & \text{if } u \in \Omega \\ \iota\left(\mathbf{X}_{u\uparrow}\right) & \text{if } u \in \Psi \\ \iota(\pi_1(\mathbf{X}_u)) & \text{if } u \in \Theta \end{cases}, \tag{7.2}$$

for $u \in \Theta$,

$$\boldsymbol{H}_u = (H_u, \preceq_u, \cdot_u, {}^{-1_u}, u) = \iota(\pi_2(\mathbf{X}_u)), \tag{7.3}$$

$\kappa = \Omega \cup \Psi \cup \Theta$, and for $u, v \in \kappa$ such that $u \leq_\kappa v$, $\varsigma_{u\to v} : G_u \to G_v$ is defined by

$$\varsigma_{u\to v} = \rho_{u\to v}|_{G_u}. \tag{7.4}$$

Call $\mathcal{G}_\mathcal{A}$ the bunch of layer groups derived from $\mathcal{A}$.

(2) Given a bunch of layer groups $\mathcal{G} = \langle \boldsymbol{G}_u, \boldsymbol{H}_u, \varsigma_{u\to v}\rangle_{\boldsymbol{\kappa}}$ with $\boldsymbol{\kappa} = \langle \Omega, \Psi, \Theta, \leq_\kappa\rangle$, $\mathcal{A}_\mathcal{G} = \langle \mathbf{X}_u, \rho_{u\to v}\rangle_{\boldsymbol{\kappa}}$ is bunch of layer algebras, called the bunch of layer algebras derived from $\mathcal{G}$, *where*

$$\mathbf{X}_u = (X_u, \leq_u, \cdot_u, \to_u, u, u'^{\mu}) = \begin{cases} \iota(\boldsymbol{G}_u) & \text{if } u \in \Omega \\ \iota(\boldsymbol{G}_u)_\downarrow & \text{if } u \in \Psi \\ Sp(\iota(\boldsymbol{G}_u), \iota(\boldsymbol{H}_u)), & \text{if } u \in \Theta \end{cases}, \tag{7.5}$$

$\kappa = \Omega \cup \Psi \cup \Theta$, and for $u, v \in \kappa$ such that $u \leq_\kappa v$, $\rho_{u\to v} : X_u \to X_v$ is defined by

$$\rho_{u\to v} = \begin{cases} \varsigma_{u\to v} & \text{if } u \notin \Theta \\ \varsigma_{u\to v} \circ h_u & \text{if } v > u \in \Theta \\ id_{X_u} & \text{if } v = u \in \Theta \end{cases}, \tag{7.6}$$

where h_u is the canonical homomorphism of $\mathbf{X}_u$.

$$\langle \boldsymbol{G}_u, \boldsymbol{H}_u, \varsigma_{u\to v} \rangle_{\langle \Omega,\Psi,\Theta,\leq_\kappa \rangle}$$

The (direct) Representation Theorem

(A) *Given an odd or an even involutive FL_e-chain $\mathbf{X} = (X, \le, \cdot, \to, t, f)$ with residual complement operation $'$,*

$$\mathcal{X}_{\mathbf{X}} = \langle \boldsymbol{G}_u, \boldsymbol{H}_u, \varsigma_{u\to v} \rangle_{\langle \Omega, \Psi, \Theta, \le_\kappa \rangle}$$

is bunch of layer groups, called the bunch of layer groups of $\mathbf{X}$, *where*

$$\kappa = \{x \to x : x \in X\} = \{u \ge t : u \text{ is idempotent}\} \text{ is ordered by } \le,$$

$$\kappa_I = \{u \in \kappa \setminus \{t\} : u' \text{ is idempotent}\},$$

$$\kappa_J = \{u \in \kappa \setminus \{t\} : u' \text{ is not idempotent}\},$$

Ω, Ψ, Θ are defined by

Ω	Ψ	Θ	
$\{t\}$	κ_J	κ_I	*if $\mathbf{X}$ is odd*
$\emptyset$	$\kappa_J \cup \{t\}$	κ_I	*if $\mathbf{X}$ is even and f is not idempotent*
$\emptyset$	κ_J	$\kappa_I \cup \{t\}$	*if $\mathbf{X}$ is even and f is idempotent*

for $u \in \kappa$,

$$\begin{array}{lcll} \boldsymbol{G}_u & = & (G_u, \le, \cdot, {}^{-1}, u) & \text{if } u \in \kappa, \\ \boldsymbol{H}_u & = & (H_u, \le, \cdot, {}^{-1}, u) & \text{if } u \in \Theta, \end{array}$$

where $X_u = \{x \in X : x \to x = u\}$, $H_u = \{x \in X_u : xu' < x\}$, $\dot{H}_u = \{xu' : x \in H_u\}$,

(8.1) $$G_u = \begin{cases} X_u & \text{if } u \notin \Theta \\ X_u \setminus H_u^\bullet & \text{if } u \in \Theta \end{cases}$$

(8.2) $$x^{-1} = x \to u,$$

and for $u, v \in \kappa$ such that $u \le v$, $\varsigma_{u\to v} : G_u \to G_v$ is defined by

(8.3) $$\varsigma_{u\to v}(x) = vx.$$

(B) *Given a bunch of layer groups $\mathcal{X} = \langle \boldsymbol{G}_u, \boldsymbol{H}_u, \varsigma_{u\to v} \rangle_{\langle \Omega, \Psi, \Theta, \le_\kappa \rangle}$ with $\boldsymbol{G}_u = (G_u, \preceq_u, \cdot_u, {}^{-1_u}, u)$*

$$\mathbf{X}_{\mathcal{X}} = (X, \le, \cdot, \to, t, t')$$

is an involutive FL_e-chain with residual complement $'$, called the involutive FL_e-chain of $\mathcal{X}$, *where $\kappa = \Omega \cup \Psi \cup \Theta$, for $u \in \kappa$,*

(8.4) $$X_u = \begin{cases} G_u & \text{if } u \notin \Theta, \\ G_u \cup H_u^\bullet & \text{if } u \in \Theta, \end{cases}$$

(where $H_u^\bullet = \{h^\bullet : h \in H_u\}$ is a copy of H_u which is disjoint from G_u),

(8.5) $$X = \dot{\bigcup}_{u\in\kappa} X_u,$$

if $u \notin \Theta$ then $\le_u = \preceq_u$, if $u \in \Theta$ then $\le_u$ extends $\preceq_u$ to X_u by letting

(8.6) $$a^\bullet <_u b^\bullet \text{ and } x <_u a^\bullet <_u y \text{ if } a, b \in H_u,\ x, y \in G_u,\ a \prec_u b,\ x \prec_u a \preceq_u y,$$

for $v \in \kappa$, $\rho_v : X \to X$ is defined by

(8.7) $$\begin{array}{lcl} \rho_v(x) & = & \begin{cases} \varsigma_{u\to v}(x) & \text{if } x \in G_u \text{ and } u <_\kappa v, \\ x & \text{if } x \in G_u \text{ and } u \ge_\kappa v, \end{cases} \\ \rho_v(\dot{x}) & = & \begin{cases} \varsigma_{u\to v}(x) & \text{if } x^\bullet \in H_u^\bullet \text{ and } \Theta \ni u <_\kappa v, \\ x^\bullet & \text{if } x^\bullet \in H_u^\bullet \text{ and } \Theta \ni u \ge_\kappa v, \end{cases} \end{array}$$

by denoting for $u, v \in \kappa$, $uv = \max_\kappa(u, v)$, for $x \in X_u$ and $y \in X_v$,

(8.8) $$x \le y \text{ iff } \rho_{uv}(x) \le_{uv} \rho_{uv}(y) \text{ except if } u >_\kappa v \text{ and } \rho_{uv}(x) = \rho_{uv}(y)^{11},$$

for $u \in \Theta$, $h_u : X_u \to G_u$,

(8.9) $$\begin{array}{ll} h_u(x) = x & \text{if } x \in G_u, \\ h_u(x^\bullet) = x & \text{if } x^\bullet \in H_u^\bullet, \end{array}$$

for $x, y \in X_u$,

(8.10) $$x \cdot_u y = \begin{cases} (h_u(x) \cdot_u h_u(y))^\bullet & \text{if } u \in \Theta,\ h_u(x)h_u(y) \in H_u \text{ and } \neg(x, y \in H_u) \\ h_u(x) \cdot_u h_u(y) & \text{if } u \in \Theta,\ h_u(x)h_u(y) \notin H_u \text{ or } x, y \in H_u \\ x \cdot_u y & \text{if } u \notin \Theta \end{cases},$$

for $x \in X_u$ and $y \in X_v$,

(8.11) $$xy = \rho_{uv}(x) \cdot_{uv} \rho_{uv}(y),$$

for $x \in X$,

(8.12) $$\begin{array}{lcl} (x^\bullet)' & = & \{\ x^{-1} \quad \text{if } u \in \Theta \text{ and } x^\bullet \in H_u^\bullet \\ x' & = & \begin{cases} (x^{-1})^\bullet & \text{if } u \in \Theta \text{ and } x \in H_u \\ x^{-1} & \text{if } u \in \Theta \text{ and } x \in G_u \setminus H_u \\ x^{-1} & \text{if } u \in \Omega \text{ and } x \in G_u \\ x^{-1}{}_\downarrow & \text{if } u \in \Psi \text{ and } x \in G_u \end{cases} \end{array},$$

for $x, y \in X$,

(8.13) $$x \to y = (xy')',$$

$\to$ is the residual operation of $\cdot$,

$$t \text{ is the least element of } \kappa,$$

(8.14) $$f \text{ is the residual complement of } t,$$

and is given by

$$t' = \begin{cases} (t^{-1})^\bullet & \text{if } u \in \Theta \\ t^{-1} & \text{if } u \in \Omega \\ t^{-1}{}_\downarrow & \text{if } u \in \Psi \end{cases}.$$

In addition,

(8.15) $$\rho_v(x) = vx \quad \text{for } v \in \kappa \text{ and } x \in X,$$

$\mathbf{X}_{\mathcal{X}}$ is odd if $t \in \Omega$, even with a non-idempotent falsum if $t \in \Psi$, and even with an idempotent falsum if $t \in \Theta$.

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[11] Alternatively, one may write the ordering on X in a lexicographic fashion by $x < y$ iff $\rho_{uv}(x) <_{uv} \rho_{uv}(y)$ or $\rho_{uv}(x) = \rho_{uv}(y)$ and $u <_\kappa v$.

The densely ordered case

If $\mathbf{X}$ is densely ordered then for $u \in \Omega$

$$\boldsymbol{H}_u = \bigcup_{\kappa \ni s <_\kappa u} \varsigma^{s \to u}(\boldsymbol{G}_s)$$

and hence the representation of $\mathbf{X}$ by layer groups can be written in a simpler form:

$$\langle \boldsymbol{G}_u, \varsigma^{u \to v} \rangle_{\langle \Omega, \Psi, \Theta, \leq_\kappa \rangle}.$$

Examples

- If $\boldsymbol{G}$ is a totally ordered abelian group then $\boldsymbol{G} = \mathcal{X}_{\mathcal{A}_{\mathcal{G}}}$ where

$$\mathcal{G} = \langle \boldsymbol{G}, \emptyset, \emptyset \rangle_{\langle \{t\}, \emptyset, \emptyset, \leq_\kappa \rangle}.$$

Denote by $\mathbb{1}$ the trivial (one-element) group.

- Totally ordered even Sugihara monoids are exactly the algebras $\mathcal{X}_{\mathcal{A}_{\mathcal{G}}}$, where

$$\mathcal{G} = \langle \mathbb{1}_u, \mathbb{1}_u, \varsigma^{u \to v} \rangle_{\langle \emptyset, \emptyset, \kappa, \leq_\kappa \rangle}.$$

- Totally ordered odd Sugihara monoids are exactly the algebras $\mathcal{X}_{\mathcal{A}_{\mathcal{G}}}$, where

$$\mathcal{G} = \langle \mathbb{1}_u, \mathbb{1}_u, \varsigma^{u \to v} \rangle_{\langle \{t\}, \emptyset, \kappa \setminus \{t\}, \leq_\kappa \rangle}.$$

- Finite partial sublex products of totally ordered abelian groups have been shown in [24] to be exactly those odd involutive FL_e-chains which have finitely many positive idempotent elements. These are exactly the algebras $\mathcal{X}_{\mathcal{A}_{\mathcal{G}}}$, where κ is finite in

$$\mathcal{G} = \langle \boldsymbol{G}_u, \boldsymbol{H}_u, \varsigma^{u \to v} \rangle_{\langle \{t\}, \kappa_J, \kappa_I, \leq_\kappa \rangle}.$$

[24] S. Jenei, The Hahn embedding theorem for a class of residuated semigroups, Studia Logica (2020) 108: 1161–1206

- Algebras which can be constructed by the involutive ordinal sum construction of [22] are exactly the algebras $\mathcal{X}_{\mathcal{A}_{\mathcal{G}}}$, where

$$\mathcal{G} = \langle \boldsymbol{G}_u, \mathbb{1}_u, \varsigma^{u \to v} \rangle_{\langle \{t\}, \emptyset, \kappa \setminus \{t\}, \leq_\kappa \rangle}.$$

[22] S. Jenei, Co-rotation, co-rotation-annihilation, and involutive ordinal sum constructions of residuated semigroups, Proceedings of the 19th International Conference on Logic for Programming, Artificial Intelligence and Reasoning, 2013, Stellenbosch, South Africa, paper 73

Summary

- A representation theorem (via a one-to-one correspondence) has been presented in a constructive manner
  for the class of all

  odd or even involutive FL_e-chains

  by means of

  bunches of layer algebras
  (certain direct systems of more specific odd or even involutive FL_e-chains)

  and the class of

  bunches of layer groups
  (certain direct systems of abelian *o*-groups)

Outlook

- The ArXiv link of the related paper along with the link to a more detailed talk on the very same subject are in the abstract

- https://sites.google.com/view/nonclassicallogicwebinar/talks

Amalgamation in classes of involutive commutative residuated lattices

- Nonclassical Logic Webinar, February 5, 2021

Thank you for your attention!